

A.F. Bermant

A Course of Mathematical Analysis 1

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A. F. BERMANT

**A COURSE OF
ATHEMATICAL ANALY**

Part I

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PREFACE TO THE SEVENTH EDITION

Certain defects became apparent in the previous edition (1950, 1951) of this "Course" as a result of experience in teaching with it for two years and of criticisms from those using it as a text-book for students at higher technical schools. The aim of the revision has been to eliminate these defects. First of all, everything was excluded that fell essentially outside the limits of the technical school higher mathematics programme and at the same time was not needed to give depth to the programme material.

The book has been considerably shortened in many places as a result of leaving out details and arguments which, though making for precision from the purely mathematical point of view, made the main subject matter more difficult to assimilate.

The arrangement of the subject matter has been simplified in a number of sections, and simpler and clearer proofs have been used (see the sections on asymptotes, Lagrange's and Cauchy's formulae, Taylor's formula, the independence of the line integral on the contour of integration, etc.).

Special attention has been paid to the literary style, and to the brevity, completeness and clarity of the exposition. The narrative part has been shortened and the material of each section has been divided up as far as possible into definitions, theorems and their proofs. The introductory remarks to the chapters have been omitted.

The statements of the most important definitions and theorems are given in heavy print, to bring the reader's attention to them. On the other hand, fine print has been chosen only for parts that can actually be omitted without serious detriment to a grasp of the essential matter. For this reason, almost all the examples, a study of which is of such importance for a grasp of mathematical analysis, appear in normal print.

Some errors and misprints which had crept in have been corrected.

None of the criticisms mentioned in the first paragraph referred to the methodological principles underlying the course. These principles in fact remain unchanged.

The creative work of the engineer, both in science and industry, requires genuine mathematical knowledge, and not merely the ability

to carry out formal mathematical operations. It is very important for the engineer to master the essence of the concepts of analysis, in order to make free use of them in solving technical problems. Because of this, it is extremely important to proceed beyond a formal, superficial study of analysis. The correct approach to this must lie in the presentation of a course in accordance with a basic scheme: practice—basic concepts of analysis—their properties (theory)—methods of working—methods of application—practice. Such a scheme makes it possible to show quite clearly the connection of mathematics with practice, to reveal the material sources of the basic concepts of analysis, and to make abundantly clear the principles of applying the theory to concrete problems.

This is a mathematical text-book and not a reference book nor a collection of methods and formulae; it shows the procedure for carrying out the various operations of analysis. It aims at developing a mathematical mode of thought and exactitude in the reader and at widening his mathematical outlook. Thus all the propositions are given with a precise statement of the conditions in which they hold, and the proofs given are complete—though naturally not developed in this book to the degree of accuracy of the theory of functions of a real variable. Omissions include certain “existence theorems”; whilst recognizing the importance and usefulness of the content and motivation of these theorems, I do not feel that it is necessary to set out their formal proofs, in view of the limitations of teaching time. Where proofs are omitted, this is indicated, and the reader is referred to some more complete guide.

The text-book has been carefully analysed by the mathematics departments of the Moscow aviation institute im. Ordzhonikidze, the Moscow higher technical school im. Bauman, the Moscow steel institute im. Stalina, the All-Soviet postal energy institute and others, and advantage has been taken of their suggestions for changes and improvements. I express my deep gratitude here to the teaching staff of all these departments. I also sincerely thank those who have communicated their criticisms to me and who have suggested necessary corrections.

A number of suggestions for improving the text were made by the editor A. Z. Ryvkin, to whom I express my thanks.

Moscow

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INTRODUCTION

1. Mathematical Analysis and its Meaning

1. “Elementary” and “higher” mathematics. The branches of mathematics, which are usually combined under the general heading of “elementary mathematics” (elementary algebra, elementary geometry, trigonometry) originated in the distant past. With few exceptions the entire system of elementary geometry in the form in which it exists today was set up in the 5th–3rd centuries B.C. The ancient Babylonians had some knowledge of algebraic transformations and the solution of equations, though the birth of algebra as a science belongs to the 8th century A.D., when the famous Arabic scholar Mohammed ben Musa laid its foundations in his treatise “Al-jabr wa'l-muqabala”, from the first word of the title of which our word “algebra” derives. As regards trigonometry, this branch of mathematics originated in connection with the astronomical investigations which were being made in ancient times; though the concepts regarding trigonometric functions and their properties were not developed until the 17th and 18th centuries.

The branches of mathematics which are combined under the general heading of “higher mathematics” were developed from studies that appeared in the 17th and 18th centuries in connection with the progress of natural science and engineering. It must be mentioned that certain ideas and methods of “higher mathematics” were foreshadowed by the great mathematician, physicist and engineer of classical times Archimedes (287–212 B.C.). “Higher mathematics” is a fairly young science.

The division of mathematics into “elementary” and “higher” is mainly conventional. There is no test by which this or that mathematical fact or theorem could be described as “elementary mathematics”. At the same time, two predominant features can be distinguished in the school course which has grown up for historical and pedagogical reasons, and which we are accustomed to call “elementary mathematics”.

The first feature consists in the fact that the *objects* of study are *constant* magnitudes and figures. Some typical problems in elemen-

tary mathematics are: given an algebraic equation, find the constant number satisfying it (the root of the equation); transform a given algebraic expression into another with the aid of the rules studied in elementary algebra; calculate the value of constant geometrical quantities (e.g. length, area, volume) or construct determinate points, curves, figures possessing required properties.

In trigonometry the variation of the trigonometric quantities is considered in relation to the variation of an angle or arc, but this is done purely descriptively, i.e. it is not based on any general theory, and these considerations do not usually serve as basis for deducing the properties of trigonometric quantities. The main problems of an elementary course of trigonometry are of the same nature as the problems of geometry and algebra: simple transformations of trigonometric expressions are studied, together with the applications of trigonometric quantities to the calculation of the elements of geometric figures.

The second feature relates to *method*. Elementary algebra and elementary geometry are constructed by independent means, separately from each other. The algebraic, or to put it more generally, the analytic method in elementary algebra has no essential connection with the synthetic method in elementary geometry. (Obviously, we are not speaking here of the elementary applications of algebraic formulae to the calculations encountered in geometry and trigonometry.) Within the limits of elementary mathematics, there is no *general* method, enabling every algebraic problem to be interpreted geometrically in a unified manner, or a geometrical problem to be translated into the language of algebra and solved analytically, i.e. with the aid of calculation. The requirements of the useful arts and economics for a more penetrating study of nature led to a body of knowledge regarding the processes and phenomena observed in the world around us. First and foremost, this was concerned with physical phenomena. But to study them from the quantitative stand-point it was necessary to create a new mathematics, which would have the power to investigate *the mutual variations of the various quantities playing a part in the phenomenon*.

Mathematical analysis*—an important branch of higher mathematics—is in fact occupied, as distinct from elementary mathematics, with *the interdependence of variable quantities*.

* Mathematical analysis is often also called the differential and integral calculus.

Mathematical analysis again differs from elementary mathematics as regards method, in that it has developed on the basis of a thorough synthesis of algebraic and geometrical methods, which was first seen in analytic geometry as developed by the famous French mathematician and philosopher Descartes (1596–1650). The idea of co-ordinates was a general principle by means of which it proved possible, on the one hand, to offer simple proofs of geometrical theorems by means of algebraic working, and on the other, due to the fact that geometrical forms can easily be visualized, to establish new analytic theorems.

It must be repeated that the division of mathematics into “elementary” and “higher” is conventional. Problems foreshadowing the ideas of higher mathematics are gradually encountered more and more in elementary mathematics.

2. Magnitudes. Variables and functional relationships. The basic concept which we encounter at every step in any domain of natural science or engineering is that of magnitude. *We mean by “magnitude” anything that can be measured and expressed by a number (or numbers).* In other words, a magnitude is an entity to which a process of measurement can be applied, in the elementary form or in mathematically perfected forms. The elementary form of measurement consists in taking as the “unit of measurement” an object of the same nature as the measured object then discovering directly how many times the “unit” is “contained” or can be “put” into the measured object. The mathematical perfection and further development of this elementary form of measurement leads to the important new concepts studied in mathematical analysis—derivatives, integrals, etc.

Magnitudes of diverse types are encountered in the concrete problems of natural science and engineering; examples of these are length, area, volume, weight, temperature, velocity, force etc. On the other hand, concrete magnitudes play no part in mathematics.

Mathematical statements and laws are formulated by abstracting from the concrete nature of the magnitudes and taking into account only their numerical values. In consequence, we consider in mathematics *magnitude in general, divorced from the physical sense which it can have.* Thus mathematical theories can be applied with equal success to the investigation of any concrete magnitudes. The generality of mathematical theories finds its expression here; this quality is also called abstractness, which is sometimes wrongly taken to imply remoteness from practice or reality. F. Engels underlined this generality of mathematics in these words: “. . . in order to be in a

position to investigate these forms (spatial—A.B.) and relationships (quantitative—A.B.) in pure form, it is necessary completely to separate them from their content, this latter being put on one side as of no significance; in this way we obtain points devoid of measurements, curves devoid of thickness and width, different a and b , x and y , constants and variables. . .”*.

Among magnitudes considered simultaneously, some usually *vary* whilst the others remain *constant*. Change or motion is the prime test of what we call a *phenomenon* or *process*. A phenomenon observed in nature or in industry is perceived by us as the variation of certain magnitudes taking part in the phenomenon due to the variation of others. For instance, if we observe a mass of gas at constant temperature, we notice that its pressure changes when its volume changes.

If we observe the fall of a body under gravity (in a vacuum), its velocity is seen to vary as its distance varies from the point at which the fall began. Similarly, the distance and velocity are seen to vary in the course of time, whereas the acceleration of the body remains constant at any instant and throughout the fall.

The study of such phenomena required the introduction into mathematics of the concept of *variable*; this was done in the epoch which saw the accomplishment of the first stage in the creation of the new mathematics—the epoch of Descartes, then of Newton and Leibniz. The introduction of variables into mathematics proved to be one of the most significant events in its history. F. Engels wrote in this connection: “The Cartesian *variable* represented a turning-point in mathematics. Thanks to this, *motion* and *dialectic* made their appearance in mathematics, and again thanks to this the *differential and integral calculus* soon became a necessity. All in all, the calculus was perfected by Newton and Leibniz, though they did not invent it”**.

A variable is a magnitude which takes different numerical values; a constant is a magnitude which preserves the same numerical value.

As already mentioned, every process may be regarded (from the quantitative point of view) as the inter-variation of certain variables. This conception leads to an idea of great importance in mathematics, that of *functional relationship*, i.e. a connection between variables.

* F. ENGELS, *Anti-Dühring*, Gost., 1952, p. 37.

** F. ENGELS, *Dialektika prirody* (*Dialectics of nature*), Gost., 1952, p. 206.

The establishment and description of the connection between the variables taking part in a given process is the prime task of the natural sciences and engineering. The law of the process is in fact defined as the *functional relationship manifested in the process and characterizing it*; we can likewise say that the relationship describes the process. Thus the functional relationship between the pressure (p) and volume (v) of a gas, consisting in the fact that $p = k/v$ (k constant) at constant temperature, expresses a law (the Boyle–Mariotte law), which is obeyed by gases in suitable conditions. The expression in words of the functional relationship: *the pressure of a gas is inversely proportional to its volume* (at constant temperature) is the usual statement of this law.

The idea of functional relationship sprang from the soil of the universal principle of causality, which began to permeate the natural and other sciences in the 17th and 18th centuries. But there is an essential difference between this principle and the mathematical idea of functional relationship. The principle of causality presupposes the distinguishing of (all or only the most important) operative causes leading to the known result, whereas a functional relationship merely gives a connection between magnitudes without presupposing that the variation of one of them is the actual cause of the variation of the other. For instance, the variation of the atmospheric temperature in the course of a day is the consequence of many causes — the variation of the wind force, of the intensity of solar radiation, of the humidity of the air, etc. A functional relationship can be established here simply between the temperature and the time of day. Although the passage of time is obviously not in itself a “cause” of the temperature variation, this functional relationship can nevertheless be valuable for a quantitative description of the process of temperature variation and in the last analysis for explaining the special features of interest to us.

The primary task of mathematical analysis is a thorough-going study of functional relationships.

3. Mathematical analysis and reality. Mathematics is used in the investigation of states and processes encountered in the various social as well as natural sciences, in fact everywhere where there is a need to consider states and processes from the quantitative stand-point. (Mathematics is also concerned with problems that are not necessarily of a quantitative kind and relate to spatial forms and relationships; such problems find virtually no place in this book.) As regards

the natural sciences and engineering, mathematics is an extremely valuable method of theoretical investigation for them and a practical tool. Without the means provided first by elementary then by higher mathematics, no scientific calculation would be possible, i.e. no serious engineering or technology would be possible without mathematics. This is due to the fact that the technological sciences are based on physics, mechanics, chemistry etc., the quantitative laws of which are expressed with the aid of functions and other concepts of mathematical analysis. Galileo remarked long ago that "the laws of nature are written in the language of mathematics," whilst Engels noted that "... an acquaintance with mathematics and natural science is necessary for a dialectic and at the same time materialistic understanding of nature"*.

It is precisely because the basic laws of physics, mechanics etc. are expressed in mathematical language that we can find theoretically, with the aid of calculations and logical arguments, the consequences of known laws, and solve the new problems posed by nature and human activity.

The quantitative laws and qualitative nature of processes are intimately inter-related, as dialectical materialism teaches. Thus mathematics is fundamentally necessary for a knowledge of any process considered in science or technology in all its manifold variety and unity. It has been truly said that mathematics is the key to the mastery of technology.

As already remarked, the ideas and methods of mathematical analysis arose inevitably in the 16th and 17th centuries, due to the requirements of the natural and technological sciences, the striking development of which was in turn called forth by the needs of a rapidly changing and widening industry: "... from the very beginning, the rise and development of the sciences was conditional upon industry"***.

We shall endeavour throughout this course to use every suitable opportunity to explain the real physical sources of the basic mathematical concepts and operations introduced, and to indicate the concrete facts and circumstances which have given birth to new mathematical theories. These theories will then be stated as far as possible in strictly mathematical terms, so as later to be able to reconsider their applications, now on a wider and higher plane.

* F. ENGELS, *Anti-Dühring*, Gost., 1952, p. 10.

** F. ENGELS, *Dialektika prirody (Dialectics of nature)*, Gost., 1952, p. 145.

And finally, the significance of the theory is discussed in practical terms on this latter level.

When developing a mathematical theory (as in general the theory of any other domain of science) one must never forget the origin of the theory; it must be remembered that the decisive criterion for its truth and value lies in a living practical proof. V. I. Lenin wrote: "*The point of view of life, of practical experience, must be the first and fundamental point of view of the theory of perception*"* (my italics — A.B.)

The great Russian mathematician P. L. Chebyshev made some interesting remarks concerning the relationship of mathematical theory to practice: "A closeness of theory to practice gives the best results, and it is not only practice that gains from this; *the sciences themselves flourish under its influence; it reveals to them new subjects for study, or new aspects in long familiar subjects* (my italics — A.B.). In spite of the high degree of development attained by the mathematical sciences as a result of the labours of the great geometers (i.e. mathematicians—A.B.) in the last three centuries, practice clearly reveals their incompleteness in many respects; it poses entirely new questions for science and thus inspires the discovery of completely new methods. If theory gains much from new applications of the old method or from new developments of it, it gains still more by the discovery of new methods, and in this case science finds its true guide in practice"** . We shall again quote F. Engels, on the empirical origins of mathematics: "Both the concept of number and that of figure stem exclusively from the external world, and did not originate from pure thought. Objects having a definite shape had to exist, and these shapes had to be subjected to comparison, before it was possible to arrive at the concept of figure"†. Apropos of this, Engels makes a judgement on the subject-matter of mathematics which, by virtue of its depth and pithiness, must be regarded as the most successful and accurate of all definitions of mathematical science:

"*Pure mathematics has for its subject-matter spatial forms and quantitative relationships of the real world*—in other words, an extremely real material. The fact that this material takes an extraordinarily

* V. I. LENIN, *Sochineniya (Papers)*, 4th ed., vol. 14, p. 130.

** P. L. CHEBYSHEV, *Cherchenie geograficheskikh kart (Geographical map drawing)*, *Izbrannye matematicheskie trudy (Collected mathematical works)*, Gost., 1946, p. 100.

† F. ENGELS, *Anti-Dühring*, Gost., 1952, p. 37.

abstract form can cast only a faint shadow over its origination in the external world"* (my italics—A.B.).

In sharp contrast with the genuinely scientific attitude of dialectical materialism we have the point of view of the idealistic philosophers, who regard science as the product of the "free" (i.e. independent of practice) creativity of the human spirit. A clear confirmation of the fact that mathematical science in fact grows out of objective reality, and that its laws and relationships correctly reflect, in the special abstract form inherent in it, the real relationships of the material world, is provided by the possibility of scientific prediction. The correctness of the deductions obtained by theoretical means (from true original propositions) with the aid of mathematics, the complete agreement of the "prediction" with reality, with what in fact occurs in the future, clearly demonstrates the validity of the definition of mathematics given by Engels.

Many remarkable examples of prediction are familiar in the history of science. We shall merely touch on two examples here; they are very expressive of the role that can be played by mathematics.

(1) The French scholar J. Leverrier (1811–1877) was occupied with problems of the motion of the planets of the solar system. He started from the laws of classical mechanics, expressed in the form of familiar functional relationships. Leverrier noticed that certain of his deductions diverged from current observations; he found also that these divergences could be eliminated if the existence were assumed of a further planet with definite mass and trajectory. On the basis of his propositions a new planet, later called Neptune, was in fact soon observed at the precise time and position which he had indicated. Thus a new world was discovered by sitting at a desk, on a piece of paper, with the aid of calculations! We now feel no surprise at the precisest knowledge of future astronomical events. This is possible of course, because the requisite mathematical methods truly reflect objective laws.

(2) The most distinguished mechanical engineer of the end of the 19th and beginning of the 20th century, the Moscow professor N. E. Zhukovskii, is the founder of the science of aviation.

He discovered by mathematical means the formulae and propositions that lie at the basis of the present-day theory of flight. In particular, Zhukovskii predicted theoretically the possibility of the "figures of higher pilotage". The first figure—"looping the loop"—

* *Loc. cit.*

was shortly afterwards accomplished by the Russian army captain P. N. Nesterov (1886–1914). Thus before “looping the loop” appeared as a physical phenomenon, it was discovered mathematically!

These examples bear witness to the enormous triumph of the mathematical knowledge of nature. At every step, in the small as well as the larger problems of science and technology, we start out with a certitude—to put the matter in essential terms—that the preliminary mathematical calculation or “design” gives a true picture of what in fact occurs. Without such certitude it would be impossible for both science and technology to exist or progress.

Mathematics is a powerful means of scientific and technical prediction.

At the same time, material reality in *all* its phenomena and interdependences is reflected only approximately in our consciousness with the aid of the various sciences, including mathematical theory. Scientific progress in fact consists in making more and more accurate our knowledge of the world. “In the theory of knowledge,” Lenin remarks, “as in all other domains of science, it is necessary to argue dialectically, i.e. not to assume that our knowledge is final and invariable, but to analyse how *knowledge* comes from *ignorance*, how incomplete, imperfect knowledge becomes more complete and more exact”*.

2. Some Historical Notes

4. Great Russian mathematicians: L. P. Euler, N. I. Lobachevskii, P. L. Chebyshev. After the differential and integral calculus had been formulated as a scientific theory in the key works of Newton and Leibniz, a continuous epoch of very rapid development of mathematics followed. In the course of more than a hundred years (from the end of the 17th to the beginning of the 19th century) there was a striking expansion both of mathematical science itself and of related fields. New results and whole new branches of learning appeared continuously as from a cornucopia, inspiring scholars to further development of mathematical theory and of methods of solution of the problems of the applied sciences. A leading role was played in this fruitful period by one of the greatest mathematicians of all time, Leonard Pavlovich Euler (1707–1783), a Swiss by birth who worked for more than 30 years in the Petersburg Academy of Sciences and linked his destiny and family for all time with Russia.

* V. I. LENIN, *Sochineniya (Papers)*, 4th ed., vol. 14, p. 91.

An idea can be gained of Euler's exceptional productivity from the size of the complete collection of his papers (still not fully edited): it will contain about 70 large volumes, including almost 800 articles; more than 650 of these were first published by the press of the Petersburg Academy of Sciences. As regards the value of Euler's work, this can easily be gauged from the simple fact that a large number of results fundamental to the various branches of natural science carry his name at the present time.

The great geometer Nikolai Ivanovich Lobachevskii (1792-1856) began his scientific work some 35-40 years after Euler. A courageous innovator in science, Lobachevskii dared to destroy the tradition of Euclidian geometry hallowed by the centuries and to create a new, non-Euclidean geometry that more deeply revealed the properties of space. At the same time, Lobachevskii's work in geometry had enormous significance for the methodology of all mathematics; it meant a return to the critical inspection of the fundamentals of science and of the large accumulation of factual material, and marked the beginning of the so-called axiomatic method, applied in the construction of mathematical disciplines. Lobachevskii was the first clearly to show the physical origin of the axioms of geometry, thus refuting the teaching of the German idealist philosopher Kant regarding their *a priori* nature.

Though few of Lobachevskii's works are concerned directly with analysis, those that are, are distinguished by the genius of a thinker who foreshadowed the path of future developments.

It is impossible not to mention, in connection with Lobachevskii's distinguished personality, his progressive pedagogical and in general elucidatory and social work, which had a great influence on the establishment of higher education in Russia.

The list of Russian names in the history of the great mathematical discoveries starts with the name of a scholar in the grand manner - N. I. Lobachevskii. And from this moment, Russian scholars more and more frequently occupy a leading position in the forefront of mathematical science; it is interesting to observe here the ever-increasing tempo of both the quantitative and in general qualitative expansion and entrenchment of Russian mathematics. This has led at the present time to great achievements in the various branches of mathematics, carrying Soviet mathematics to one of the leading positions in the world. Among other characteristic features in the development of Russian, then later of Soviet mathematics, must be mentioned a proclivity for obtaining effective results, a close contact

with the applied sciences and a conviction of the possibility of finding, even if gradually, the solutions of problems no matter how difficult.

Academician Mikhail Vasil'evich Ostrogradskii (1801–1861), a great and gifted mathematician in another field, analysis, lived and worked at about the same time as Lobachevskii. A large number of discoveries are due to Ostrogradskii in the different fields of mechanics, mathematical analysis, mathematical physics, algebra, and the theory of numbers. Many of these discoveries genuinely advanced science; they formed a starting-point for other scholars and fairly soon became classical, entering into almost all the relevant world scientific literature.

The great mathematician Academician Pafnutii L'vovich Chebyshev (1821–1894) began his scientific and pedagogical work in Moscow, then continued in Petersburg. He was the founder of a great Russian mathematical school (called the "Petersburg"), progenitor of the present Soviet schools that are successfully developing Chebyshev's remarkable ideas. His studies, exceptional for their originality, presented the solutions both of problems put forward by himself and of problems which had resisted the attempts of other distinguished mathematicians. These studies were further marked by simplicity and completeness of conception. A versatile mathematician, Chebyshev also dealt successfully with problems of the applied sciences. He recognized clearly the progressive and fertile interaction of mathematical theory and practice, which he in fact expressed in the fine words which we quoted above (Sec. 3). One of Chebyshev's outstanding works was concerned with establishing and investigating the new problem of approximating functions by polynomials (see Chapter IV). This work, which arose in connection with his investigation of certain purely engineering problems (in the theory of mechanisms), revealed an entire new trend in mathematical analysis. To Chebyshev is also due the construction of the best calculating machine of his time.

Chebyshev is widely accepted as one of those scholars who set the trend of development of science for many decades by their work.

Chebyshev's pupils included the famous mechanical scientist and mathematician A. M. Lyapunov (1857–1918), the distinguished mathematician A. A. Markov (1856–1922), and many other great Russian scholars whom it is impossible for us to mention here. We shall merely further recall Chebyshev's outstanding contemporary S. V. Kovalevskii (1850–1891), the first woman to be a professor of

mathematics, whose achievements in science (chiefly in mathematical analysis) compelled the Petersburg Academy of Sciences to break with tradition and elect a woman as corresponding member.

The full flowering of mathematics and its applications to the natural sciences and engineering began in the USSR after the White October revolution. The scale of scientific work entirely changed and became incommensurable with that of previous times; the number of scholars increased by many times; the interests of science came to be regarded as the interests of the state. All this reacted in the most favourable manner on the position of mathematics, which achieved remarkable successes. The great discoveries of Soviet scholars relate to mathematical fields which go far beyond the bounds of the theory expounded in this book.

5. Leading Russian applied mathematicians: N. E. Zhukovskii, S. A. Chaplygin, A. N. Krylov. This text-book is essentially suited to future engineers, so that we should like to refer the reader again to the exceptional importance of the tradition of Russian science, which does not divorce theory from practice, but is guided in practice by the indications of theory. We need to be inspired by the examples of the great mathematicians Euler and Chebyshev, who solved difficult engineering problems, and of the famous engineers and mechanical scientists, N. E. Zhukovskii and A. N. Krylov, who solved difficult mathematical problems. The former saw concrete reality beyond their abstract mathematical constructions, and very often took engineering problems as their starting-point in these constructions; the latter were always orientated towards truth in their engineering investigations by mathematical theory, in which they also had a large stake.

We have already mentioned N. E. Zhukovskii (1847–1922), “the father of Russian aviation”. We must add here that he was concerned not only with aviation but with other domains of technology and mechanics. The important results that he obtained were reduced by him to a form suitable for practical investigations. All his investigations are marked by superior mathematical learning. He was also responsible for some interesting mathematical works.

Zhukovskii’s pupil, Academician and Hero of Soviet Labour S. A. Chaplygin (1869–1942) was one of the greatest mechanical scientists of recent times; he continued brilliantly his teacher’s work in the field of aeronautics and shared with him the glory of organizing a first-class Soviet school in hydro- and aerodynamics and a number

of leading Soviet scientific and technical institutes. Chaplygin possessed to perfection the art of devising subtle mathematical methods for the solution of complex technical problems.

The Academician and Hero of Soviet Labour A. N. Krylov (1863–1945), a highly talented engineer and gifted mathematician, was a living example of the harmonious union in one individual of theory with practice which should distinguish a specialist in applied science; he must be regarded as an example to be emulated.

A. N. Krylov was a leading naval architect, an expert on applied mathematics and approximate methods in mathematics, the designer and first constructor of a complex mathematical machine, an historian and gifted pedagogue, and finally, a brilliant interpreter and popularizer of science and literature. To Krylov is due the notable Russian translation of Newton's classical work "*Principia Mathematica*". Krylov furnished this translation with extremely valuable commentaries.

3. Real numbers

6. Real numbers. The real axis. Numbers make their appearance as a result of *counting* (the "natural numbers," 1, 2, 3, . . .) and as a result of the *measurement of magnitudes*. In mathematical analysis we are interested only in the numerical values of magnitudes—we operate in it only with *numbers*; it is thus necessary to start the study of analysis with a consideration of the set of so-called *real numbers*.

Numbers are divided into *rational* and *irrational*. We describe as rational, numbers of the form p/q , where p and q are integers. We can quote as examples of irrational numbers (already familiar from a course of "elementary mathematics"): $\sqrt{2}$, $\sqrt{3}$, $\log_{10} 3$, π , $\sin 20^\circ$ and so on. The set of all rational and irrational numbers forms the set of real numbers.

Let us now turn to the geometrical interpretation of numbers. We take a straight line and some point O on it, which we take as the initial point or origin for reckoning lengths. We choose a scale (i.e. a segment is taken as the unit of length) and mark off from point O segments whose lengths are expressed by rational numbers in our scale. We shall imagine the straight line as horizontal; segments marked off to the right of point O will be reckoned to correspond to positive numbers, and segments to the left to negative

numbers. This establishes a positive direction on the line—from left to right*.

A straight line on which are indicated an origin for reckoning lengths, a scale and a positive direction is called a numeral or real axis.

Suppose that the end of the segment, whose length in the chosen scale is equal to the rational number a , falls at the point M of the line. We then say that the point M represents the number a and that the number a is the *co-ordinate* of point M . Obviously, the point O represents the number zero.

The points of a real axis representing rational numbers are termed rational points. The rational points lie on the real line very thickly, or as we say, are *everywhere dense*. To be precise, this means that, no matter how small a segment we take of the real axis, we can find within it as many rational numbers as we please.

This can be shown as follows. Any segment of the real axis is included between some two points A and B with rational (say integral) co-ordinates a and b . The mid-point C of the segment between points A and B will again be a rational point, because its co-ordinate is equal to the number $\frac{1}{2}(a + b)$, which is rational. By the same argument, the mid-point C_1 of the segment AC and the mid-point C_2 of the segment CB are similarly rational points. On continuing to bisect every time all the segments obtained, we shall always arrive at new rational points. The distance between two successive points thus obtained after the n th bisection of segment AB will be equal to the length of segment AB divided by 2^n , as may readily be seen. Hence it follows that this distance can be made as small as desired with sufficiently large n . This means that, no matter how small the initial segment, in the long run at least two rational points fall in it from the number of those which we get by means of our bisection of segment AB . It may be seen on the basis of the same argument that as many rational points as desired lie between these two rational points and therefore inside the given segment.

Nevertheless, rational points do not exhaust all the points of the real axis; there are other points on it, the irrational. To indicate just one irrational point, we mark off a segment to the right of O , equal to the diagonal of the square whose sides are equal to unity. The end

* The opposite direction—from right to left—can be chosen as the positive direction on the line. But the positive direction on a horizontal axis is usually taken as we have taken it, from left to right.

of this segment falls at a point which is not rational, because, as is shown in elementary mathematics, its length is not expressed by any rational number. Every point lying at a rational distance from the irrational point indicated is also an irrational point.

Thus there are "as many" of these irrational points on the line as there are rational points, in other words, they are distributed everywhere densely on the line; but we can indicate as many other irrational points as desired*, and each of them generates, as above, "as many" new irrational points as there are rational points.

Hence it follows that it is impossible to accomplish the co-ordinate principle on a straight line with the aid of only the rational numbers; all the points of the line, whilst they are geometrically of the same value, would not be of the same value analytically: some would have their specific numerical characteristics, i.e. co-ordinates, whilst others — including the great majority — would have no co-ordinates.

A logical demand arises for the introduction of new numbers which will play the same role for irrational points as is played by rational numbers for rational points, i.e. they will be their co-ordinates. These numbers are called irrational, like the corresponding points of the real axis. We shall not expound here the theory in which it is shown how, starting from the rational numbers, the irrational numbers can be defined, and how the ordinary rules of the arithmetical operations are then extended to them. This theory is only of purely mathematical interest. It is important to emphasize here merely that there is a one-to-one correspondence between the system of all points of the real axis and the set of all real numbers: *every point of the real axis represents a single number (rational or irrational), and conversely, every number (rational or irrational) is the co-ordinate of a single point of the real axis.*

The various relationships between numbers are represented in a very obvious manner on our real axis. For instance, it is evident that a point corresponding to a lesser number (in the algebraic sense) lies to the left of a point representing a larger number; that in view of this a point corresponding to a number between two given numbers, lies between the points representing these numbers; that the difference between two numbers is represented by the segment between the

* For instance, if we mark off to the right of O segments equal to the diagonals of rectangles with sides equal to 1 and 2, 1 and 3, 1 and 4 and so on, the ends of these segments represent irrational points.

corresponding points, directed to the right if the difference is positive, and to the left if negative, and so on.

By virtue of this correspondence between numbers and points of the real axis, it is sometimes convenient not to distinguish between the concepts of "point" and "number", and to understand a "number" by a point or a point by a "number".

As already mentioned, there are as many rational points as desired on any segment, however small, of the real axis, i.e. in particular, in any neighbourhood of a given irrational point. Consequently, an irrational number can be expressed with any degree of accuracy* with the aid of rational numbers. Properly speaking, we get our actual picture of an irrational number from the rational numbers which, whilst differing slightly from it, serve as its approximate values. It is evident from this that rational numbers alone would be quite sufficient for practical calculations; irrational numbers are also necessary, however, for strict theoretical arguments, because only then is the number system complete, in the same sense in which the set of all points of the real axis is complete when there are no exclusions, e.g. we are not dealing only with the set of rational points.

7. Intervals. Absolute values. We now introduce some terms and concepts which will be of use later.

I. Definition. An interval is the set of all numbers (points) included between two numbers (points) called the ends of the interval.

The interval with ends $x = a$ and $x = b$, where $a < b$, can be specified by the inequalities $a < x < b$; it is also written as (a, b) . If, in addition to the set of points of the interval, we consider its ends, a closed interval is obtained. The closed interval with ends $x = a$ and $x = b$ ($a < b$) is given by the inequalities $a \leq x \leq b$; it is written as $[a, b]$. The interval (a, b) is said to be open, or non-closed**, in contrast with the closed interval. If one end is associated with the interval and the other is not, an interval is obtained which is open at one side, or semi-open. A semi-open interval is given by the inequalities $a \leq x < b$, if the end a is included, and by $a < x \leq b$ if the end $x = b$ is included; they are written respectively as $[a, b)$ and $(a, b]$ ***.

* The ideas of "accuracy" and "approximate value" should be familiar to the reader. We shall define them below (Sec. 8).

** A closed interval is often called a *segment* in mathematical literature, the term "interval" being preserved for open intervals only.

*** Thus inclusion of an end in an interval is always denoted by a square bracket, and exclusion by a curved bracket.

We shall simply speak of an interval when there is no need to distinguish whether or not the ends are included in it.

In addition to finite intervals, infinite intervals are often encountered, i.e. either the set of all numbers less than (or not greater than) some number c , or the set of all numbers greater than (or not less than) the number c , or finally, the set of all real numbers. These infinite intervals are given respectively by the inequalities $-\infty < x < c$ (or $-\infty < x \leq c$), $c < x < +\infty$ (or $c \leq x < +\infty$) or $-\infty < x < +\infty$. They are written as $(-\infty, c)$ (or $(-\infty, c]$), $(c, +\infty)$ (or $[c, +\infty)$) and $(-\infty, +\infty)$.

Definition. The interval of length $2l$ with centre at the point a is called an l -neighbourhood of point a .

The concept of the neighbourhood of a point is often used in what follows.

II. Let us consider the concept of the absolute value of a number or expression.

Definition. The absolute value (or modulus) $|A|$ of a number (or expression) A is the number (or expression), equal to A itself if A is positive or zero, and equal to $-A$ if A is negative:

$$|A| = \begin{cases} A, & \text{if } A \geq 0, \\ -A, & \text{if } A < 0. \end{cases}$$

Thus $|A|$ is always a non-negative number (positive or zero).

On the real axis, $|A|$ is the length of the segment from the origin to the point representing A .

We have by definition:

$$-|A| \leq A \leq |A|;$$

for, if $A > 0$, we have the sign of equality on the right and the sign of inequality on the left (the positive number A is greater than the negative number $-A$), whilst if $A < 0$, we have conversely the sign of inequality on the right (the negative number A is less than the positive number $|A|$), and the sign of equality on the left.

We also remark that

$$|A| = \sqrt{A^2}.$$

The relationship

$$|A| \leq B, \text{ where } B > 0, \quad (*)$$

is equivalent to the two relationships

$$-B \leq A \leq B. \quad (**)$$

For, relationship (*) implies that point A is at a distance not greater than B from the point zero; and this is possible only when the point A lies in the interval $[-B, B]$, i.e. belongs to the B -neighbourhood of the point zero, in other words, when relationship (**) holds; the converse is also obvious.

Let us prove the following two propositions on absolute values:

(1) *The absolute value of a sum is not greater than the sum of the absolute values of the terms, i.e.*

$$|A + B + C + \dots + D| \leq |A| + |B| + |C| + \dots + |D| \quad (***)$$

We shall first take the case of two terms. We have:

$$-|A| \leq A \leq |A| \quad \text{and} \quad -|B| \leq B \leq |B|;$$

adding term by term gives us

$$-(|A| + |B|) \leq A + B \leq (|A| + |B|),$$

and these two relationships are equivalent to the stated relationship

$$|A + B| \leq |A| + |B|.$$

On passing successively from two to three, from three to four etc. terms, our proposition is readily proved in general. The equality can only hold in relationship (***) when all the terms have the same sign.

(2) *The absolute value of a difference is not less than the difference of the absolute values of minuend and subtrahend:*

$$|A - B| \geq |A| - |B|.$$

For the proof, we put $A - B = C$; then $A = B + C$ and by what has been shown:

$$|A| = |B + C| \leq |B| + |C| = |B| + |A - B|,$$

i.e.

$$|A - B| \geq |A| - |B|.$$

As regards a product and quotient, by virtue of the definitions of these operations, *the absolute value of a product is equal to the product of the absolute values of the factors:*

$$|A \cdot B \cdot C \dots D| = |A| \cdot |B| \cdot |C| \dots |D|,$$

and the absolute value of a quotient is equal to the quotient of the absolute values of dividend and divisor:

$$\left| \frac{A}{B} \right| = \frac{|A|}{|B|}.$$

8. A note on approximations. In technology and practical life we are almost always concerned with numbers expressing the approximate values of magnitudes.

It is a familiar fact that measurements of the same magnitude, carried out in turn, lead to numbers which differ from each other, even though ever so slightly. When we say that the measure of a concrete magnitude is equal to a given number, we imply that numbers are obtained as a result of our measurements which differ so little from the given number that the difference can be neglected.

The applications of mathematical analysis to the natural sciences and engineering are necessarily accompanied by and finished off by calculations. We must therefore learn to make sound calculations! But what meaning is to be attached to the idea of a "sound calculation"?

The *first test* of a sound calculation consists in its sufficient "accuracy". The *second test* should be taken as the opposite quality in the ordinary sense: a calculation should be carried as far as the limits of useful accuracy, without excessive accuracy. Finally, the *third test* is the speed of carrying out the operations, the economy in labour and time.

Calculations should always be made in accordance with these three requirements.

We shall assume that the reader is acquainted with the elementary rules and methods of approximations relating to the arithmetical operations.* The calculations encountered in mathematical analysis are generally reduced to the carrying out of *arithmetical operations* only.

We shall now define the basic concepts of approximations.

We shall take A to mean the accurate value of some magnitude, or simply some number.

* See this "Course", 6th ed. (1950, 1951); see also: A. N. KRYLOV, *Lectures on approximations* (*Lektsii o priblizhennykh vychisleniyakh*), Izd. Akad. Nauk, 1933; YA. BEZIKOVICH, *Approximations* (*Priblizhennye vychisleniya*), Gost., 1949; G. SANDEN, *Elements of applied analysis*, GTTI, 1932.

Definition. The number a is called an approximate value of A with the error α if

$$A - a = \alpha.$$

The sign $\approx$ is used to denote an approximate equality:

$$A \approx a.$$

The number a may be either less than A , $a < A$, or greater than A , $a > A$. In the first case $\alpha > 0$ and a is a low approximation of A , in the second case $\alpha < 0$ and a is a high approximation of A . It is most commonly a matter of indifference which of these cases holds.

The error α , like the accurate value A , is usually unknown, but the limit δ becomes known, which the error is found not to exceed (in absolute value): the number $|\alpha|$ does not exceed δ , i.e. $|\alpha| \leq \delta$.

Definition. If the absolute value of the error $|\alpha|$ of the approximate value a does not exceed the positive number δ , this number is called the *maximum (absolute) error* of the approximate value:

$$|A - a| \leq \delta,$$

or

$$-\delta \leq A - a \leq \delta, \delta > 0.$$

The number δ is not uniquely defined: any number greater than an already known maximum error can itself be taken as the maximum error, i.e. as the number δ .

For instance, 3.24 is the approximate value of the number 3.2447 with the error 0.0047, or of 3.2447... with the absolute maximum error 0.005, since $0.0047 \dots < 0.005$. However, even when the error is known, it is often more convenient to take instead of it the maximum error; in the first example it is convenient to indicate the maximum error 0.005 instead of the error 0.0047.

The following statements are generally useful:

a is the approximate value of A with the (absolute) error α or with the maximum (absolute) error δ ;

a is the approximate value of A to an accuracy of δ ; the approximate equality $A \approx a$ has the error α or the maximum error δ , or again, holds to an accuracy of δ ;

the accuracy of the approximate value (or equality) is the greater, the smaller δ .

We shall understand by the term "approximation" both the process of finding increasingly more accurate approximate values and the approximate values themselves. We say that a is an "approximation" of A .

It is important to notice that nothing can be said about the *quality* of an approximation from the (*absolute*) error or maximum (*absolute*) error alone, since we also need to know what the approximated value itself is. The approximate values of two different quantities may have the same absolute (or maximum absolute) error, yet it is quite obvious that the better of these approximate values is the one which is larger (in absolute value). For instance, let the absolute maximum error in determining both the height of a 25-storey block and the height of a 2-storey house be equal to 1 *m*. It is clear that we get a better picture of the height of the block than of the house, since the error committed in the first case is a smaller fraction of the actual height than in the second case. We judge the quality of an approximation by the fraction which the absolute or maximum absolute error represents of the approximate value.

Definition. The ratio α' of the (*absolute*) error α to the approximate value a is termed the *relative error*: $\alpha' = \alpha/a$.

The ratio δ' of the (*absolute*) maximum error δ to the approximate value a is termed the *maximum relative error*: $\delta' = \delta/a$.

It is quite obvious that two approximations can only be compared by means of the relative, and not the absolute, maximum errors.

The smaller the relative error α' (or maximum relative error δ'), the better or more accurate the approximation.

Relative maximum or ordinary errors are usually given in percentages (i.e. in parts of 100) or sometimes in parts of 1000. To obtain the maximum error in per cent or in parts per 1000, the relative maximum error must be multiplied by 100 or 1000 respectively.

We say that: a is the approximate value of A with a maximum relative error δ' ; the approximate equality, $A \approx a$, has a maximum relative error δ' .

An accuracy in units of per cent is quite sufficient in practical calculations and it is only carried to tenths of a per cent in special cases of exceptional accuracy.

CHAPTER I

FUNCTIONS

1. Functions and Methods of Specifying Them

9. The concept of function.

Definition. A magnitude is called a *function* of a second (variable) magnitude if, for any given value of the second magnitude, there is a corresponding definite value (or values) of the first.

We can also say that the magnitudes are connected by a *functional relationship*.

The variable magnitude whose value we can choose at discretion is called the *independent variable*; the function may also be called the *dependent variable*. Once the values of the independent variable have been chosen, the function cannot be given arbitrary values—it will have strictly determined values, corresponding to the chosen values of the independent variable. The values of the function depend on the values taken for the independent variable, and as a rule they change as the latter change. This explains the terms “independent” and “dependent” variables.

Only real magnitudes will be considered in this “Course”, unless there is a special proviso to the contrary.

Examples. 1. The radius r of a circle and its area s can take different values. But the radius and area depend on one another; in fact, for each value of the radius there is a corresponding definite value of the area: the area is equal to the square of the radius multiplied by the constant number π :

$$s = \pi r^2.$$

2. The distance s traversed by a body falling freely in a vacuum and the time t of descent are variables. They depend on each other. This dependence is expressed by the law of free fall: the distance traversed (in the absence of an initial velocity) is equal to the square

of time multiplied by a constant number—half the acceleration due to gravity:

$$s = \frac{1}{2} g t^2.$$

Attention should be paid to the fact that the dependence of the area of a circle on the radius and the dependence of the distance on the time of free fall are the same: in both our examples, the function is proportional to the square of the independent variable. This alone is of importance from the mathematical point of view: what interests us in mathematics is how the function depends on the independent variable, and we are not at all concerned in what concrete quantities are concealed behind these variables. From this point of view, the above functions are examples of the same so-called *quadratic function* (see Sec. 18).

It is not necessary, in the definition of a function, that the function should vary when the independent variable varies. The only thing that matters is that *there should be definite values of the function corresponding to every given value of the independent variable*. We can therefore regard as a function a magnitude which does not vary at all, i.e. a *constant*.

This point of view can also be arrived at as follows: a magnitude depending on some variable magnitude, and in general changing with it, can happen to be constant with particular assumptions. There is naturally no advantage in distinguishing the particular case from the general, i.e. in regarding our magnitude as no longer a function in the particular case. For instance,

$$\frac{ab}{a + b \tan^2 x} + c \sin^2 x,$$

where a, b, c are constants, depends on the variable x , i.e. is a function of it. At the same time, in the particular case when $a = b = c$, this value is equal to the constant c identically (i.e. for any x).

Thus, *a constant can be regarded as a function, viz. as the function whose values are equal to one another for all values of the independent variable*.

10. Methods of specifying functions. To specify a function means to indicate the set of all values taken by the independent variable, and the means by which we find the values of the function corresponding to these values of the independent variable. A function can be specified in various ways: the most important of these are by *tables*, by a *formula* and by a *graph*.

I. In the tabular specification of a function we simply write down the series of values of the independent variable and the corresponding values of the function. This method is often employed. Tables of logarithms, i.e. of the logarithmic function, tables of the trigonometric functions and their logarithms, etc. are well known.

The tabular method of specifying functions is specially prevalent in the natural sciences and engineering. The numerical results of successive observations in an experiment are usually grouped in the form of tables.

For example, the following table was obtained when studying the dependence of the electrical resistance r of a copper rod on its temperature t :

t	19.1	25.0	30.1	36.0	40.0	45.1	50.0
r	76.30	77.80	79.75	80.80	82.35	83.90	85.10

The resistance is a function of the temperature, and the above table gives the values of this function for the indicated values of the independent variable.

The defect of specifying a function by a table lies in the fact that it is usually impossible to specify the function completely by this means: values of the independent variable exist which are not contained in the table. For instance, the table above does not allow us to answer the question as to what the resistance is at temperatures less than 19.1° or greater than 50.0° . Similarly, we cannot know from the table the resistance at temperatures 24.2° and 37.43° , which are simply not included among the temperature values*.

Another defect of tables, especially apparent if they are large ones, is that they do not provide a clear visual picture. More often than not, it is difficult to see from a table the nature of the variation of the function with variation of the independent variable.

The advantage of tables for specifying functions is that, for any value of the independent variable contained in the table, we can find at once the corresponding value of the function without any measurements and calculations.

* Special methods exist enabling us to find approximately, in certain cases, the values of the function corresponding to values of the independent variable not included in the table.

II. Before turning to the second method of specifying functions, we note that:

An analytic expression is *a set of familiar mathematical operations** carried out in a definite sequence on numbers and variables.

The fundamental method of specifying a function is by means of a formula, or as is sometimes said, analytically. It consists in writing down formulae, in the form of equations between analytic expressions, in which two variables take part; from the values of one variable, taken as the independent variable, we determine by calculation the corresponding values of the other—the function.

For example, the formula

$$y = \frac{(3x^5 - \sqrt{2x-1}) \tan(2x+3)}{\log(1+x) - \sin x}$$

defines y directly as a function of the variable x . On substituting a given value of x on the right-hand side and carrying out the operations indicated, we obtain the corresponding value of y .

If y is given as a function of the independent variable x by means of formulae, these are said to establish the functional relationship between x and y , or alternatively, to connect x and y .

The advantages of the analytic method of specifying functions include:

(a) brevity and compactness: short formulae define the values of the function for all the considered values of the independent variable;

(b) the possibility of calculating the value of the function for any value of the independent variable for which the operations of the formula have a meaning**;

(c) finally, and this is the most important, the possibility of using the apparatus of mathematical analysis when investigating the function, this apparatus being best adapted precisely to the analytic form of specifying a function.

If, for instance, we want to investigate by the methods of mathematical analysis the functional relationship between the temperature and electrical resistance of a copper rod, we must have a formula

* The following elementary operations are included here: addition, subtraction, multiplication, raising to a power, extracting a root, taking logarithms, and the trigonometric and inverse trigonometric operations.

** For example, in the formula quoted above it is impossible to find the value of function y at $x = 0$, because the denominator of the right-hand side then vanishes and the formula loses its meaning.

connecting the resistance and temperature, and not separate (even if numerous) corresponding values of the function and independent variable.

The disadvantages of the analytic specification of a function are:

- (a) insufficient ease of visualization;
- (b) the need for making calculations, which are sometimes very laborious.

III. We turn finally to the specification of a function by a graph.

Definition. The **graph** of a function (in a system of rectangular Cartesian co-ordinates) is the locus of points whose abscissae are the values of the independent variable, and ordinates the corresponding values of the function.

In other words, if we take the abscissa equal to some value of the independent variable (on a definite scale), the ordinate of the corresponding point of the graph (also measured on a definite, though in general different scale) must be equal to the value of the function corresponding to this value of the independent variable.

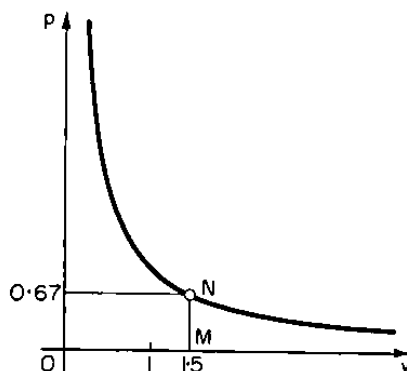


FIG. 1

Figure 1 shows the graph of the dependence of the pressure of 1.293 kg of air on its volume at 0°C . On the axis of abscissae, 1 cm represents one cubic metre of air, whilst 1 cm on the axis of ordinates represents a pressure of one atmosphere. In order to find with the aid of this graph the value of the function corresponding to a given value of the independent variable (say $v = 1.5$ cm), we have to mark off a segment on the axis of abscissae representing the value of the independent variable on the chosen scale ($OM = 1.5$ cm), and measure the ordinate of the point of the curve corresponding to this abscissa. In our present case $p = 0.67$ atm, since the segment MN is equal to 0.67 cm.

Graphs of functions usually consist of curves (and in particular, straight lines). Conversely, every curve on the co-ordinate plane represents some function—the function whose value is equal to the ordinate of the curve at the value of the independent variable equal to the corresponding abscissa.

The concepts of curve and function are thus closely related. A curve (its graph) is generated by the specification of a function; a function is generated by the specification of a curve in the co-ordinate plane—the function whose graph the curve is. We say in this latter case that the function is specified graphically.

If a function is given by a formula, connecting the independent variable x and the function y , the graph of the function is a curve, the equation of which is the equation of the formula. Conversely, a given curve is the graph of the function which is defined by the equation of the curve.

Thus the graph of the quadratic function $y = x^2$ is the parabola whose axis coincides with the positive semi-axis of ordinates, since the formula $y = x^2$ is in fact the equation of this parabola; the graph of the function $y = 1/x$ is a rectangular hyperbola for which the co-ordinate axes are asymptotes, since the equation $xy = 1$ is the equation of this hyperbola.

The graph of a constant is obviously a straight line parallel to the axis of abscissae.

Functions are quite often specified graphically in physics and technology. This happens for instance with recording devices, which automatically record the variation of one magnitude with another (most commonly, time). A curve is obtained as a result on the paper of the device, which specifies graphically the recorded function. Such devices include e.g. the barograph, tracing a barogram—a graph of the atmospheric pressure, and the thermograph, tracing a thermogram—a graph of the temperature, as a function of time.

A graph is sometimes the only possible means (or the simplest means) of specifying a function. As with tables, the apparatus of mathematical analysis cannot be directly or accurately applied to graphs, though they have, along with this serious defect, the supreme advantage of being visual; it is this which makes them exceptionally useful in the investigation of functions.

2. Notation for and Classification of Functions

11. Notation. Many magnitudes have well-established symbols. Thus time (and temperature) are most commonly denoted by the letter t , mass by m , velocity by v , and so on. In mathematics, however, we

are concerned with variables independently of their physical meaning; they are usually denoted by $x, y, z, u, \dots$

We often speak, especially in theoretical discussions, of a functional relationship between two variables, say x and y , without indicating precisely what it is; and instead of saying in words: " y is a function of the variable x ", we write symbolically, for brevity:

$$y = f(x).$$

The letter f stands for the word "function".

The independent variable is put in brackets after the symbol for the functional relationship. This notation is read as follows: " y is a function of x ", or more briefly, " y equals $f(x)$ ".

We see that the letter f no longer denotes a magnitude, but a relationship, i.e. the law of correspondence between the independent variable and the function. Other letters as well as f may be used to symbolize the functional relationship: F, φ, Φ etc.

We write $y = f(x)$, not only when it does not matter what precise function we are talking about, but also in cases when we have in mind some definite but unknown function. Even if the function is known, we still often write it symbolically, for the sake of brevity in working that involves it.

Just as the familiar symbols $\log, \sin, \cos, \tan, \arcsin$ and so on denote functions defined by well-known elementary operations performed on the independent variable, the general symbol f indicates, in the analytic specification of a function, the set of operations and their sequence, which have to be performed on x in order to obtain y . For instance, the symbol f in the equation

$$f(x) = \sqrt{x^2 + \sin x}$$

denotes that the square and sine of a given x are found, the results added and the square root of the sum extracted.

If we want to express the fact that the circumference l and area s of a circle are functions of its radius r , we could write $l = f(r)$ and $s = f(r)$. But the circumference does not depend on the radius in the same way as the area. When simultaneously considering the two functions, the difference between them should be reflected in the writing. It is better to write here:

$$l = f(r) \quad \text{and} \quad s = F(r),$$

or

$$l = f_1(r) \quad \text{and} \quad s = f_2(r)$$

and so on.

The converse may hold in other cases: two pairs of different magnitudes may be connected by the same functional relationship. Thus the area s of a circle has the same relationship to its radius r as the surface of a sphere Q to its diameter d . To reflect this fact symbolically, we write:

$$s = f(r) \quad \text{and} \quad Q = f(d),$$

using the same symbol for the function in both cases.

A particular value of a function is found by substituting the relevant value of the independent variable in the expression for the function. Let the particular value of the function $y = f(x)$ at $x = a$ be equal to b . This is written as follows:

$$b = f(a).$$

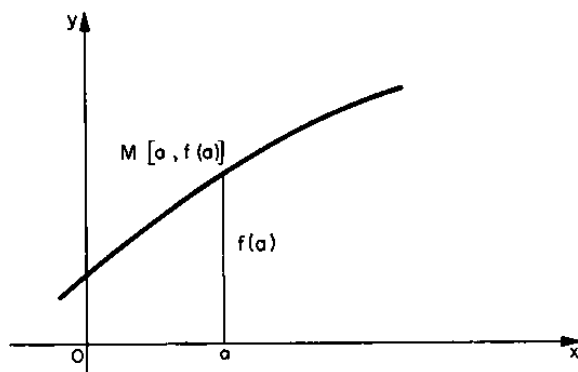


FIG. 2

The latter equality cannot be read like the previous one: “ b is a function of a ”, because both a and b are constants here. It should be read as: “ b is equal to the value of function f at a value of the independent variable equal to a ”, though it is usually read more briefly as “ b equals fa ”, bearing in mind the meaning to be attached to these words.

The point M of the graph of the function $y = f(x)$, corresponding to the abscissa $x = a$, has the ordinate $f(a)$, this fact being written as $M[a, f(a)]$ (Fig. 2).

12. Function of a function. Elementary functions.

I. Cases are quite often encountered when y is expressed as a function of z :

$$y = \varphi(z),$$

whilst z is in turn a function of a variable x :

$$z = \psi(x).$$

In this case y is a function of x , expressed by means of the variable z , here called the intermediate variable. In fact, one or several definite values of z correspond to a given value of x , and one or several definite values of y to each of these values of z ; all in all, one or several definite values of y correspond to the given value of x , i.e. y is a function of x . The expression for it can be written as follows:

$$y = \varphi[\psi(x)].$$

Definition. The variable from the values of which we can find all the possible values of the function, is termed in general the argument of the function.

The argument of a function may be the independent variable or may be an intermediate variable, i.e. it is also a function of the independent variable.

The specification of a magnitude y as a function of a magnitude x by means of a chain of two functions is said to define a *function of a function*. The function $y = \varphi[\psi(x)]$ is a function of a function of the independent variable x .

For example, $y = (\sin x)^2$ is a function of a function of x ; y is a quadratic function of an argument which is, in turn, a familiar trigonometric function (the sine) of the independent variable x ; on writing z for the intermediate variable, we can write: $y = z^2$, $z = \sin x$.

Let us take as a second example the kinetic energy j of a body, of mass m , thrown vertically upwards with initial velocity v_0 .

As we know,

$$j = \frac{mv^2}{2} = \varphi(v),$$

i.e. the kinetic energy is a function of the velocity of the body. But this velocity is a function of time. If we neglect the air resistance, we have

$$v = v_0 - gt = \psi(t),$$

where g is the acceleration due to gravity.

The kinetic energy of the body can therefore be considered as a function of a function of time:

$$j = \varphi[\psi(t)] = \frac{m[\psi(t)]^2}{2} = \frac{m(v_0 - gt)^2}{2}.$$

The chain of functions with the aid of which a function of a function is formed can be made up of as many links as desired (more than

two). It is often useful to be able to "split up" a given function of a function into the individual links.

II. The functions used in a general course of mathematical analysis, and in the applied sciences, are usually represented in the form of functions of a function formed from an extremely limited number of elementary functions, called the basic elementary functions.

Definition. The basic elementary functions are the following:

- (1) the power function: $y = x^n$, where n is a real number;
- (2) the exponential function: $y = a^x$, where a is a positive number different from 1;
- (3) the logarithmic function: $y = \log_a x$, where the base of logarithms a is a positive number different from 1;
- (4) the trigonometric functions: $y = \sin x$, $y = \cos x$, $y = \tan x$, and also the more rarely used: $y = \cot x$, $y = \sec x$, $y = \operatorname{cosec} x$;
- (5) the inverse trigonometric functions: $y = \arcsin x$, $y = \arccos x$, $y = \arctan x$, and also $y = \operatorname{arccot} x$, $y = \operatorname{arcsec} x$, $y = \operatorname{arccosec} x$.

We shall consider these functions in arts. 5 and 6.

The basic elementary functions are so to speak the "bricks" from which other functions, and in particular the so-called "elementary functions", are constructed.

Definition. An elementary function is a function which can be specified by analytic expressions, formed only from the basic elementary functions and constants, with the aid of a finite number of arithmetical operations (addition, subtraction, multiplication and division) and a finite number of operations consisting of taking a function of a function.

For instance, the functions given by the following formulae are elementary functions:

$$y = \frac{2 + \sqrt[3]{x}}{3 - \sqrt{x}}, \quad y = \log(x + \sqrt{1 + x^2}), \quad y = \arctan \sqrt{\frac{1 + \sin x}{1 - \sin x}}$$

and so on.

Examples of non-elementary functions are given in the next section. We shall be concerned almost exclusively with elementary functions in this book.

13. The classification of functions. We shall establish here the division of functions into explicit and implicit, algebraic and transcendental, single-valued and many-valued.

I. EXPLICIT AND IMPLICIT FUNCTIONS. The division of functions into explicit and implicit refers only to functions specified analytically.

Definition. A function y of an argument x is described as *explicit* if it is specified by quoting directly the analytic expression in x which is equal to function y .

We can say alternatively, that an explicit function is specified by an equation between y (the function) and x (the argument), which is solved with respect to y .

For instance, we can say that the functions given at the end of Sec. 12 are explicit.

But an equation between two variables, which is not solved for either of them, can define one as a function of the other.

For instance, in the equation of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the ordinate y can be regarded as a function of the abscissa x . For there are two values of y , which can be found from the equation of the ellipse, corresponding to every value of x lying between $-a$ and $+a$. This function is specified *implicitly*. To pass to its explicit specification, we only need to solve the equation for y :

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2}.$$

This transition is not always easy, however, and is sometimes quite impossible (if only a finite number of basic elementary functions is used). Thus the equation

$$y^3 - 3axy + x^3 = 0$$

defines y as a function of x , but it is more difficult to express this function in the explicit form than in the first example, since to do so we have to solve an equation of the third degree. Whereas the function y , defined by the equation

$$xy - 2^x + 2^y = 0,$$

cannot in fact be specified in the explicit form by means of elementary functions, because the given equation cannot be solved for y with the aid of a finite number of arithmetical operations and basic elementary functions. Nevertheless, this equation does define y as a

function of x . For, suppose we assign to x the value $x = x_0$ and substitute this in the left-hand side:

$$x_0 y - 2^{x_0} + 2^y = 0.$$

On solving this numerical equation with respect to y , we obtain a value $y = y_0$, which is regarded as corresponding to the value $x = x_0$.

Our definition of implicit function will apply, independently of whether or not y can be found from the given equation in the form of an elementary function of x .

Definition. An *implicit function* y of an argument x is a function, defined by an equation connecting x and y , and not solved with respect to y^* .

II. ALGEBRAIC AND TRANSCENDENTAL FUNCTIONS. We specially distinguish, among functions specified analytically, those which are defined with the aid of the so-called algebraic operations: addition, subtraction, multiplication, division and raising to a power with rational index.

Definition. A function whose value can be obtained by carrying out a finite number of algebraic operations on the independent variable is called an *explicit algebraic function*.

An example of an algebraic function is

$$y = \frac{\sqrt{x-1}}{x^2+1}. \quad (\text{A})$$

If we exclude extracting a root from the operations mentioned, we get the definition of a rational function.

Definition. A function is *rational* if its value can be obtained by carrying out a finite number of additions, subtractions, multiplications and divisions on the independent variable.

An example of a rational function is

$$y = \frac{\frac{3}{x^2} - 2x + \frac{1}{x} - 5}{x - \frac{2}{x^2} + \frac{1}{x^2} - 1}. \quad (\text{B})$$

* It must not be thought, however, that any equation between x and y necessarily defines a function of one variable. For instance, the equation $x^2 + y^2 + 1 = 0$ cannot be satisfied for any real values of x and y (a sum of positive numbers cannot be equal to zero!), i.e. it does not define a function.

Definition. An algebraic function which is not rational is called **irrational**.

Thus function (A) is irrational.

Definition. A rational function is described as **integral** if its definition does not require division by an expression containing the independent variable.

Examples of integral rational functions are

$$y = \left(\frac{3(x-2)^3 + 5}{7} - \frac{x}{a} \right)^2, \quad y = ax^2 + bx + c.$$

Definition. Integral rational functions are called **polynomials** (a single term is regarded as a particular case of a polynomial).

A polynomial can always be arranged in decreasing (or increasing) powers of the independent variable, i.e. written as the sum

$$y = P(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n,$$

where $a_0, a_1, \dots, a_{n-1}, a_n$ are constant coefficients.

Definition. A rational function is called a **rational fraction** if it is represented by a fraction, the numerator and denominator of which are polynomials.

Every rational function $R(x)$ can be written with the aid of identical transformations as the quotient of two polynomials: $R(x) = P(x)/Q(x)$, i.e. as a rational fraction. For instance, let us write function (B) as a rational fraction:

$$y = \frac{\frac{3}{x^2} - 2x + \frac{1}{x} - 5}{x - \frac{2}{x^3} + \frac{1}{x^2} - 1} = \frac{-2x^4 - 5x^3 + x^2 + 3x}{x^4 - x^3 + x - 2}.$$

The above definition of explicit algebraic function is narrower than the general definition now accepted in mathematics. According to this general definition, an algebraic function is a function y which is a root of an algebraic equation whose coefficients are polynomials in the independent variable x :

$$P_0(x)y^n + P_1(x)y^{n-1} + \dots + P_{n-1}(x)y + P_n(x) = 0.$$

It can be shown that every explicit algebraic function is also algebraic in the sense of the general definition. For instance, function (A) is a root of the equation

$$(x^4 + 2x^2 + 1)y^2 + (2x^2 + 2)y + (-x + 1) = 0. \quad (*)$$

There exist algebraic equations, however, which cannot be solved with the aid of a finite number of algebraic operations. Thus the functions whose values can

be calculated with the aid of a finite number of algebraic operations do not exhaust the set of all functions which are algebraic in the sense of the general definition.

Definition. A function which is not algebraic is called *transcendental*.

Examples of transcendental functions are

$$y = \sin x, \quad y = x 2^x, \quad y = \arctan \frac{x}{1+x}.$$

III. SINGLE-VALUED AND MANY-VALUED FUNCTIONS

Definition. A function is called *single-valued* if one definite value of the function corresponds to each considered value of the independent variable. Otherwise the function is called *many-valued*.

For example,

$$y = x^3, \quad y = \sin x, \quad y = \log x$$

are single-valued functions, whilst $y = \pm \sqrt{x}$ is a many-valued, in fact two-valued, function.

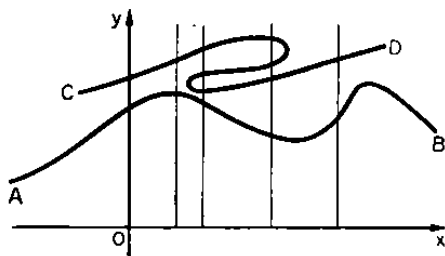


FIG. 3

The fact that $y = f(x)$ is a single-valued function is expressed geometrically as follows: every straight line, parallel to Oy and cutting the graph of the function, has only one point in common with it (curve AB in Fig. 3). The graph of a many-valued function is cut by certain straight lines parallel to Oy in more than one point (curve CD in Fig. 3). Thus in the case of the function $y = \pm \sqrt{x}$, a straight line parallel to the axis of ordinates and lying to the right of it intersects the graph of the function in two points; one point, above Ox , defines one value of the function (the positive value), the other below Ox , the other (negative) value.

The part of the graph of a many-valued function, cutting any straight line parallel to Oy in not more than one point, defines a

single-valued function. This isolated function is called the single-valued branch of the many-valued function*.

If, for instance, we take the upper half of the parabola, we get the graph of the function $y = \sqrt{x}$, which is the single-valued branch of the two-valued function $y = \pm \sqrt{x}$, this branch being positive for all positive x .

We shall always be concerned with single-valued functions, unless there is a special proviso to the contrary.

3. Elementary Investigation of Functions

14. Domain of definition of a function. Domain of definiteness of an analytic expression.

I. It was by no means assumed, in the definition of function given in Sec. 1, that the independent variable must take all real values. On the contrary, any previously chosen set of numbers can be taken as the domain of variation of the independent variable; if we relate a definite value of another magnitude to each of these numbers in accordance with some rule, the second magnitude thereby becomes a function of the first.

A function is said to be defined for a given value of the argument (at a given point), if it acquires a corresponding value (or values, if the function is many-valued) in accordance with some rule.

Definition. The *domain of definition (or of existence)* of a function is the set of all points of the real axis at which it is defined.

For each point of the domain of definition of a function there are one or more corresponding points of the graph representing the function; at every point of the axis where the function is not defined, there is no corresponding point of the graph.

In the definition of function given earlier (Sec. 9), the values forming the domain of definition of a function were called the "considered values of the independent variable".

In analysis we usually consider functions whose domains of definition are of one of the following two types:

(1) the set of positive "integral" points of the real axis, i.e. the points $x = 1, x = 2, x = 3, \dots$ (or in general the points $x = N, x = N + 1, x = N + 2, \dots$, where N is any positive integer);

* A part of the graph is usually isolated which consists of a continuous curve (see Sec. 35).

(2) one or more intervals of the real axis.

We can take as an example of a function with a domain of definition of the first type the perimeter P_n of a regular polygon, inscribed in a given circle. For, P_n depends only on the integral number n of sides of the polygon.

Definition. A function $y = f(x)$, in which the independent variable runs through the positive integers: $x = 1, 2, 3, \dots$, is termed a **function of an integral argument**.

The values

$$f(1), f(2), f(3), \dots,$$

taken by a function of an integral argument form a sequence, i.e. a set which is enumerated with the aid of the series of integers and is arranged in order of increasing number.

Conversely, every sequence is the set of values of some function of an integral argument.

For, given the sequence $u_1, u_2, u_3, \dots$, we thereby establish a corresponding value $f(n) = u_n$ for each positive integer n . For example, the terms of the geometrical progression $1/2, 1/4, 1/8, \dots$ are consecutive values of the function $f(n) = 1/2^n$, $n = 1, 2, \dots$

The graph of a function of an integral argument (or of a sequence) is a set of separate points on the plane with integral abscissae and corresponding ordinates. An example of a function with a domain of definition of the second type is

$$\tau = f(t),$$

where τ is the atmospheric temperature measured during a day, and t is time, measured say in hours. The domain of definition of this function is the interval $[0, 24]$.

II. Definition. The **domain of definiteness of an analytic expression** is the set of all points of the real axis at which this expression has a definite (real) value.

For example, the domain of definiteness of the expression $\sqrt{1-x^2}$ is the closed interval $[-1, +1]$, since the expression has a definite real value for any x satisfying the inequalities $-1 \leq x \leq +1$, whilst it does not have a real value at a point lying outside the interval $[-1, +1]$, i.e. for $|x| > 1$.

The domain of definiteness of the expression $1/x$ is the entire real axis except for the point $x = 0$, at which this expression loses

its meaning, because it is impossible to divide by zero; hence the domain of definiteness consists of two open infinite intervals:

$$-\infty < x < 0 \quad \text{and} \quad 0 < x < +\infty.$$

The domain of definiteness of any polynomial is the entire real axis.

The domain of definiteness of the expression $\sqrt{x^2 - 1}$ is the set of two infinite intervals $-\infty < x \leq -1$ and $+1 \leq x < +\infty$ (or alternatively, $|x| \geq 1$), i.e. the entire real axis except for the interval $(-1, +1)$.

The domain of definiteness of $\log x$ is the open infinite interval $0 < x < \infty$ ($\log x$ is not defined for $x \leq 0$, since real logarithms of negative numbers and zero do not exist).

The domain of definiteness of the expression $\frac{2x^2 - \log(x+5)}{\sqrt{8-x^3}}$ is the open interval $(-5, 2)$, since $\log(x+5)$ is not defined for $x \leq -5$, $\sqrt{8-x^3}$ is not defined for $x > 2$, and the entire fraction is not defined for $x = 2$, because the denominator vanishes.

III. *If $y = f(x)$ is an elementary function, specified without any supplementary conditions, it is always understood that its domain of definition (existence) is the domain of definiteness of the analytic expression that specifies it.*

Thus the domain of definition of the function $y = \sqrt{1-x^2}$ is the closed interval $[-1, +1]$; the domain of definition of the function $y = 1/x$ is the entire axis, except for the point $x = 0$, and so on.

A function should not be confused with the analytic expression that specifies it. An analytically specified function can be defined so that, in one interval of values of the independent variable, it is given by one analytic expression, and in another interval by quite a different expression. For instance, let us specify a function $y = f(x)$ on the positive semi-axis Ox as:

$$f(x) = 2\sqrt{x}, \quad \text{if } 0 \leq x \leq 1,$$

$$f(x) = 1 + x, \quad \text{if } x > 1.$$

This will be the function, completely defined for all $x \geq 0$, whose graph is shown in Fig. 4; but the function is represented in the intervals $[0, 1]$ and $[1, \infty]$ by two different analytic expressions, which are not reducible one to the other. In accordance with our terminology, the function is non-elementary in the interval $[0, \infty)$.

Let us take a further example of a non-elementary function. We shall specify the function $y = f(x)$ as: equal to $x \sin(1/x)$ for any x different from zero, and equal to zero for $x = 0$:

$$f(x) = x \sin \frac{1}{x} \quad (x \neq 0) \quad \text{and} \quad f(0) = 0.$$

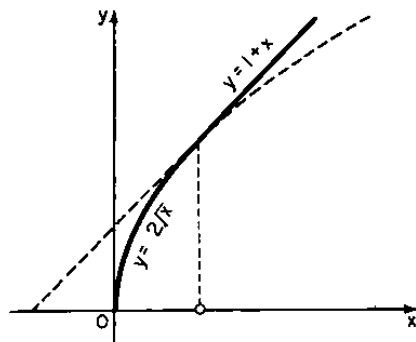


FIG. 4

Here the function is represented by the same expression throughout Ox , except for the point $x = 0$, at which the value of the function is specially defined. The expression $x \sin(1/x)$ is not defined at $x = 0$ (the argument of the sine loses its meaning), whereas the domain of existence of $f(x)$ is the entire axis Ox .

On the other hand, we shall encounter in the second part of this "Course" (Chapter XIV) cases when two different analytic (infinite) expressions define the same function in one interval and different functions in another.

Concrete problems often lead to functions written as analytic expressions which are defined in a wider domain than the sense of the problem permits. For instance, the area s of a circle, as a function of its radius r :

$$s = \pi r^2,$$

is represented by an expression whose domain of definiteness is the entire real axis, though there is no sense here in giving r negative values; the domain of definition of the function is only the positive semi-axis of the argument.

15. Elements of the behaviour of functions. To investigate a function means to characterize the course of its variation (or we also say, its behaviour) when the independent variable varies. We shall

always assume here (unless there is a proviso to the contrary) that the *independent variable increases, passing from smaller to larger values through all intermediate values.*

Thus, if we are speaking of a function of an integral argument, the argument takes integral values consecutively; whereas if we are investigating a function defined throughout some interval, the independent variable will run from left to right through this interval, without omitting any of its points.

Definition. An independent variable is said to be *continuous* if, when taking two different values, it takes all intermediate values.

A function which is specified at all points of an interval is termed a *function of a continuous argument.*

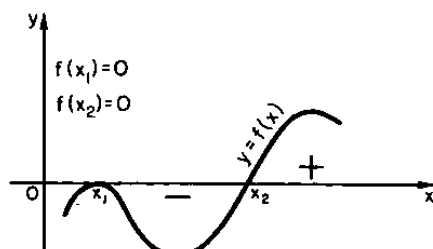


FIG. 5

As we extend our means of investigation, we shall gradually be able to give an increasingly complete description of the behaviour of a function. Since we have at present only the means of elementary mathematics and analytic geometry, we shall limit our characterization here of the behaviour of a function to the following elementary features: (I) the sign in a given interval; (II) oddness or evenness; (III) periodicity; (IV) growth in a given interval.

I. Definition. An interval in which function $y = f(x)$ retains the same sign (+ or -) is called an *interval of constancy of sign* of the function. A value x for which the function vanishes, $f(x) = 0$, is called a *zero* (or *root*) of the function.

In an interval of positive sign of a function, its graph is located above Ox , whilst it is below Ox in an interval of negative sign; at a zero of the function, the graph has a common point with Ox (Fig. 5).

II. Definition. A function $y = f(x)$, defined in an interval $(-a, a)$, is described as *even* if, on changing the sign of any value of the argu-

ment belonging to this interval, the value of the function is unchanged :

$$f(-x) = f(x).$$

A function $y = f(x)$, defined in an interval $(-a, a)$, is described as **odd** if, on changing the sign of any value of the argument belonging to this interval, only the sign (and not the absolute value) of the function is changed :

$$f(-x) = -f(x).$$

Examples of even functions are $y = x^2$, $y = \cos x$, whilst $y = x^3$, $y = \sin x$ are examples of odd functions.

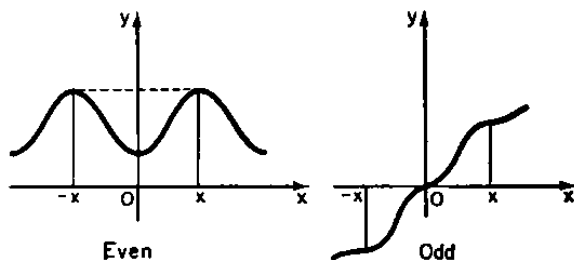


FIG. 6

The graph of an even function is symmetrical with respect to Oy ; the graph of an odd function is symmetrical with respect to the origin. (We leave the reader to prove this; see Fig. 6.)

Obviously, a function may be neither even nor odd; for example, $y = x + 1$, $y = 2 \sin x + 3 \cos x$, $y = 2^x$ and so on are such functions.

III. *Definition.* A function $y = f(x)$ is said to be **periodic** if there exists a constant number $a \neq 0$ such that, on adding it to any value of the argument, the value of the function is unchanged :

$$f(x + a) = f(x).$$

If a function is periodic, the following equalities also hold:

$$f(x + 2a) = f(x), \quad f(x + 3a) = f(x),$$

$$f(x - a) = f(x), \quad f(x - 2a) = f(x)$$

and in general:

$$f(x + ka) = f(x)$$

for any x and for any integer k (positive or negative).

Definition. The *period* of a function is the least positive number which, when added to any value of the argument, leaves the function unchanged.

An example of a periodic function is $y = \sin x$; its period is equal to 2π . (Sometimes any number (not zero) which, when added to any value of the argument, leaves the value of the function unchanged, is called the period, whilst the least such positive number is called the basic period.)

It is sufficient to consider the behaviour of a periodic function in any interval whose length is equal to the period, e.g. in the interval $0 \leq x \leq a$, where a is the period; the values of the function at other points of Ox are got simply by repetition of its values in this interval. The graph of a periodic function is got by repetition of the part of the graph, corresponding to an interval of Ox equal in length to the period (see Fig. 7).

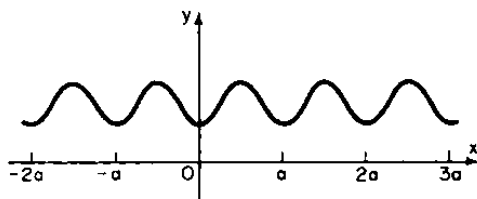


FIG. 7

IV. *Increase* and *decrease* are very important features in the behaviour of a function.

Definition. A function is said to be *increasing* in an interval if greater values of the function correspond to greater values of the independent variable in the interval*; it is said to be *decreasing* if smaller values of the function correspond to larger values of the independent variable.

Thus $f(x)$ is increasing in the interval (a, b) if, for all values x_1, x_2 satisfying the condition $a < x_1 < x_2 < b$, we have $f(x_1) < f(x_2)$, whilst it is decreasing if, for all x_1, x_2 satisfying the condition stated, we have $f(x_1) > f(x_2)$.

If we consider the graph of a function from left to right (which corresponds to increase of the argument x), the point of the graph

* When we speak of the variation (increase, decrease) of a variable, we have in mind algebraic variation. If we are speaking of variation of the absolute value of the variable, this is specially mentioned.

moves upwards for an increasing function (curve AB of Fig. 8), and moves downwards for a decreasing function (curve CD in Fig. 8).

An interval of the independent variable in which the function is increasing is called an interval of increase of the function, whilst an interval in which the function is decreasing is an interval of decrease. A function is described as monotonic in both an interval of increase and an interval of decrease.

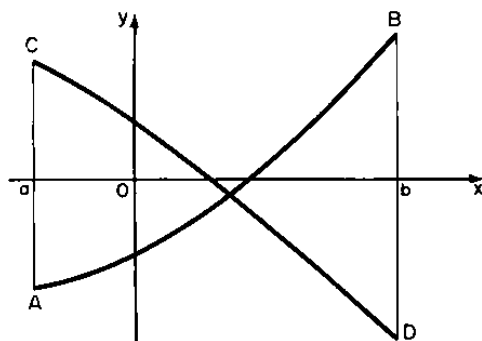


FIG. 8

The value of a function which is greater or less than all its other values in an interval is called the maximum or minimum value respectively of the function in the interval.

Investigation of the four features described here often enables a fairly clear picture to be obtained of the behaviour of a function.

If the function is elementary and specified without any additional conditions, the first task of an investigation of the function is to find its domain of definition (i.e. the domain of definiteness of the analytic expression specifying it).

16. Graphical investigation of a function. Linear combinations of functions. When the graph of a function is known, its behaviour can be seen from direct inspection of the figure. We can find from the figure the zeros of the function and the intervals in which it is monotonic, or in which it has constant sign; we can also discover whether it is an even or odd function, or neither, and whether it has a period*.

For instance, the graph of the function $y = f(x)$ illustrated in Fig. 9 shows that $f(x)$ vanishes at the points x_0, x_2, x_4 : $f(x_0) = 0$,

* The accuracy is limited, however, by the accuracy of the figure and of the measurements which have to be made on it.

$f(x_2) = 0$, $f(x_4) = 0$; that it is negative in the intervals (a, x_0) and (x_2, x_4) , and positive in the intervals (x_0, x_2) and (x_4, b) ; that it is increasing in the intervals (a, x_1) and (x_3, x_5) , and decreasing in the intervals (x_1, x_3) and (x_5, b) ; and finally, that the maximum of the function throughout the interval (a, b) is attained at the point x_5 , and the minimum at the point x_3 .

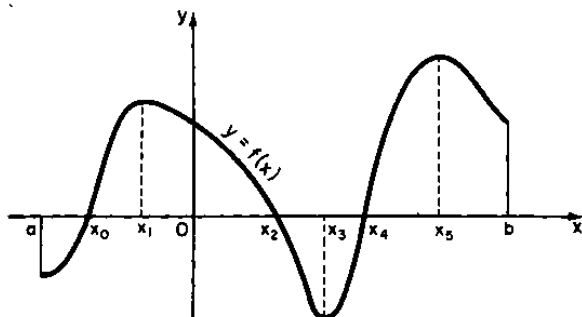


FIG. 9

For the present, we can draw the graph of a function either on the basis of familiar geometrical properties of the curve representing the function, or by plotting points. This latter method consists in choosing any number n of values of the independent variable: $x_1, x_2, \dots, x_n$, calculating the corresponding values of the function: $f(x_1), f(x_2), \dots, f(x_n)$, and thus finding the n points: $M_1(x_1, f(x_1))$, $M_2(x_2, f(x_2))$, $\dots$, $M_n(x_n, f(x_n))$, belonging to the graph of the function. On drawing a curve by eye through these points, we get an approximate representation of the graph, this being generally speaking the more accurate, the greater the number n , i.e. the more points M are plotted on the co-ordinate plane.

If we can draw the graph of the function $y = f(x)$ with any choice of scales on the axes, we can easily obtain the graph of the function $y = a_1 f(a_2 x)$, where a_1 and a_2 are any real constants (naturally, different from zero). To do this we have to bring in new scales on the axes, connected with the previous ones by the formulae $x' = a_2 x$ and $y' = y/a_1$ (in the case of negative a_1 or a_2 , we also have to change the direction on the axis of ordinates or axis of abscissae); the graph of the function $y' = f(x')$, drawn on the new scales, yields on the old scales precisely the graph of the function $y = a_1 f(a_2 x)$. This transformation of the co-ordinates is called a "scale transformation". On associating with it a parallel displacement of the co-ordinate

system, we can also draw the graph of the function $y = a_1 f(a_2 x + a_3) + a_4$, where a_1, a_2, a_3, a_4 are any real constants.

Thus if we know a method of drawing the graph of the function $y = f(x)$ with arbitrary scales on the axes, it is easy also to draw the graph say of the function $y = 2f(2x + 1)$. On putting $y' = \frac{1}{2}y$ (which implies a doubling of the scale on the axis of ordinates),

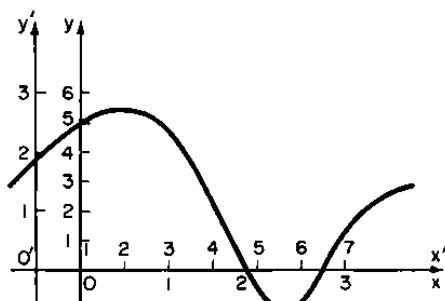


FIG. 10

$x' = 2x$ (which implies a halving of the scale on the axis of abscissae), then $x'' = x' + 1$ (which displaces the co-ordinate system to the left along the axis of abscissae by 1 unit in the new scale), we get $y' = f(x'')$. It only remains to trace the graph of our function in the system of axes $O'x''y'$, to obtain the curve which is the required graph in our original system of axes Oxy (Fig. 10; see the example in Sec. 25).

Definition. The function given by the equation

$$y = a_1 f_1(x) + a_2 f_2(x) + \cdots + a_n f_n(x),$$

where $a_1, a_2, \dots, a_n$ are constants, is called a linear combination of functions $f_1(x), f_2(x), \dots, f_n(x)$.

Knowing the graphs of the functions $y = f_1(x), y = f_2(x), \dots, y = f_n(x)$, the graph of a linear combination of the functions can be drawn by purely geometrical means. We multiply the ordinates of the graphs of each of the functions $f_1(x), f_2(x), \dots, f_n(x)$ by the corresponding constants $a_1, a_2, \dots, a_n$ and add the ordinates of the curves obtained.

Thus if we know the graphs of certain functions, the functions resulting from the above simple combinations of them can be investigated graphically.

It must be mentioned, however, that the peculiarities of an analytically specified function are not discovered in the majority of cases

from its graph, but vice versa, i.e. the graph is drawn on the basis of peculiarities discovered by analytic means. Various branches of mathematical analysis are used for this purpose, primarily the differential calculus.

4. Elementary Functions

17. Direct proportionality and linear functions. Increments.

Definition. A direct proportionality is a relationship expressed by the formula

$$y = ax,$$

where a is a constant.

Its characteristic feature is, that when one variable changes, the other changes in the same ratio, in other words, if x_1 and y_1 , x_2 and y_2 , are two pairs of corresponding values of x and y , and if $x_2 = kx_1$, then also $y_2 = ky_1$. For we have: $y_1 = ax_1$, $y_2 = ax_2$; but if $x_2 = kx_1$, then $y_2 = akx_1 = kax_1$, i.e. $y_2 = ky_1$.

This property is stated in words as: y is proportional to x . The constant a is called the *coefficient of proportionality*.

A direct proportionality is a particular case of the relationship established by a linear function.

Definition. A linear function is a function of the form

$$y = ax + b,$$

where a , b are constants.

This yields a direct proportionality when $b = 0^*$.

A linear function is defined throughout Ox . As we know from analytic geometry, its graph is a straight line. The constant coefficients a and b are respectively equal to the slope of the straight line (i.e. the tangent of the angle between the straight line and Ox) and its initial ordinate (i.e. the ordinate of the point of the line at $x = 0$). When $b = 0$ the graph of a linear function is a straight line through the origin.

It is clear from this geometrical interpretation of a linear function that it is increasing throughout Ox when a is positive, and decreasing if a is negative.

This feature of a linear function is also easily explained analytically.

* The reader will have no difficulty in showing that, if $b \neq 0$, y is not proportional to x .

We shall first introduce a simple but important concept, and also a notation which will be continually employed in what follows.

Definition. Let a magnitude u pass from one value u_1 (the initial value) to another value u_2 (the final value). The difference between these values is called the increment of u ; it is denoted by Δu , i.e.

$$\Delta u = u_2 - u_1.$$

The increment Δu can be either positive or negative; in the first case the variable increases on passing from u_1 to $u_2 = u_1 + \Delta u$, and in the second it decreases. If $\Delta u = 0$, then $u_2 = u_1$ and u does not vary.

It must be emphasized that Δu does not denote the product of some magnitude Δ and the variable u , but is a unified symbol, denoting the difference between the incremented value of the variable and its initial value.

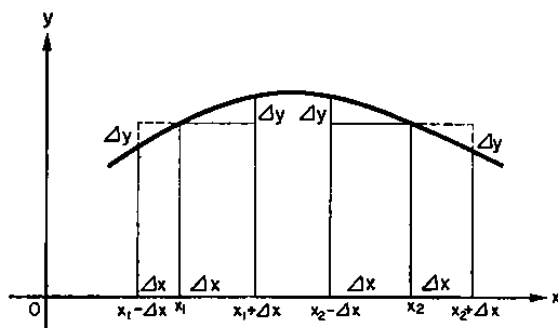


FIG. 11

When the independent variable x receives an increment Δx , a function of it $y = f(x)$ also receives an increment: $\Delta y = f(x + \Delta x) - f(x)$. The increment of the function is represented on its graph by the segment of the corresponding ordinate, taken with the + sign or the - depending on whether the ordinate increases or decreases when the abscissa passes from the initial to the final position (Fig. 11).

Generally speaking, the increment of a function depends both on the initial value of the argument x , and on the increment Δx given to x .

We now prove the following important property of a linear function.

THEOREM. The increment of a linear function is proportional to the increment of the argument and does not depend on the initial value of the latter.

Proof. Let the argument receive the increment Δx , passing from the value x to the value $x + \Delta x$; then the function receives some increment Δy , passing from the value y to the value $y + \Delta y$. We have:

$$y = ax + b$$

and

$$y + \Delta y = a(x + \Delta x) + b.$$

On subtracting the first equation from the second, we find:

$$\Delta y = a\Delta x,$$

which is what we wished to prove.

This relationship is also easily deduced from geometrical considerations. It is clear from Fig. 12 that Δy and Δx are adjacent sides of the right-angled triangle ABC , where $\tan \alpha = a$. Thus

$$\Delta y = \tan \alpha \cdot \Delta x = a\Delta x.$$

This property completely characterizes a linear function; in other words, there are no other functions which have this property.

CONVERSE. If the increment of a function is proportional to the increment of the argument and does not depend on the initial value of the argument, the function is linear.

Proof. Let the increment Δy of the function $y = f(x)$ be always (i.e. for any x) proportional to the increment of the independent variable Δx :

$$\Delta y = a\Delta x,$$

where a is a constant. We shall show that $f(x)$ is a linear function. Let $f(x)$ be equal to b for $x = 0$, i.e. $f(0) = b$. We now give the argument some arbitrary new value x ; the function now takes the new value $y = f(x)$. The increment of the argument is $\Delta x = x - 0 = x$, whilst the increment of the function is $\Delta y = y - b$. Hence, by hypothesis, we must have $y - b = ax$, i.e.

$$y = ax + b$$

for any x .

This is what we had to prove.

Let us take a function which is not linear, say $y = x^2$. Firstly, let the initial value of the argument be $x = 4$ and $\Delta x = 2$; then

$$\Delta y = (x + \Delta x)^2 - x^2 = 6^2 - 4^2 = 20.$$

Further, let $x = 5$ be the initial value of the argument, and as before let $\Delta x = 2$; then

$$\Delta y = 7^2 - 5^2 = 24.$$

We have obtained different increments of the function with equal increments of the argument. The increment of the function $y = x^2$ therefore depends not only on the increment of the argument but also on its initial value.

In general, the dependence of the increment of the function on the increment of the argument is fairly complicated, and its investigation is an important problem of analysis, which we shall discuss in the following chapters. The linear function is an exception in this respect: its increment depends only on the increment of the argument and is also directly proportional to it. This fact gives the linear function its special simplicity.

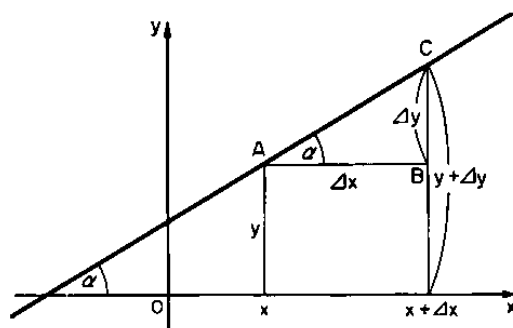


FIG. 12

We shall now use the relationship $\Delta y = a\Delta x$ for an analytic investigation of the linear function. We shall assume that $\Delta x > 0$; the sign of Δy now coincides with that of a , so that, with $a > 0$, we also have $\Delta y > 0$, i.e. the function is increasing, whilst with $a < 0$, $\Delta y < 0$, and the function is decreasing. We have arrived at the result which was obtained earlier by geometrical arguments.

Any *uniform process* is described by a linear function.

Definition. A process (involving two magnitudes x and y) is said to be uniform if there correspond, to any two equal increments of one magnitude (x), equal increments of the other magnitude (y).

It can be shown that the equation $\Delta y = a\Delta x$, where a is a constant coefficient, holds for a uniform process*. It follows from this that y is a linear function of x .

* Strictly speaking, we need to suppose further, for our statement to be valid, that the process proceeds "continuously", i.e. that the required functional relationship possesses the property of "continuity". The question of the continuity of a function will be discussed in Chapter II, art. 3.

For example, a motion is uniform if equal paths (s) are traversed in equal intervals of time (t); then $\Delta s = v \Delta t$, where v is the constant velocity. We have from this, by virtue of the property of a linear function: $s = vt + s_0$, where s_0 is the path traversed up to the instant $t = 0$. This is called the equation of uniform motion.

Uniform processes are the simplest of all and are at the same time extremely important.

18. Quadratic functions.

Definition. A quadratic function is a second degree polynomial:

$$y = ax^2 + bx + c,$$

where a , b , c are constant coefficients.

A quadratic function is defined throughout the real axis.

In the simplest case, when $b = c = 0$, the function has the form $y = ax^2$; its graph is a parabola with vertex at the origin and axis directed along the axis of ordinates. Thus the parabola axis coincides with the positive semi-axis Oy , when $a > 0$, and with the negative semi-axis Oy , when $a < 0$. In the general case the graph of the function $y = ax^2 + bx + c$ is, as we know from analytic geometry, a parabola whose axis is parallel to Oy , whose vertex lies at the point $\left(-b/2a, \frac{-b^2 - 4ac}{4a}\right)$, and whose parameter is equal to $1/2a$. The branches of the parabola are directed upwards or downwards depending on whether $a > 0$ or $a < 0$ (Fig. 13).

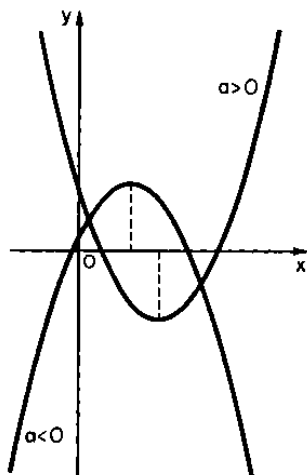


FIG. 13

The behaviour of a quadratic function is more complicated than that of a linear function. Whereas a linear function is monotonic throughout Ox , a quadratic function changes the nature of its variation once; it is either decreasing at first, then increasing, or vice versa. To be exact, if $a > 0$, the function is first decreasing, in the interval $(-\infty, -b/2a)$, attains its minimum value $y = -(b^2 - 4ac)/4a$, then is increasing in the interval $(-b/2a, +\infty)$; it never attains a maximum value: however large a number we assign, a point x can be indicated for which y is still larger. Conversely, if $a < 0$, the function is increasing at first in the interval $(-\infty, -b/2a)$, attains its maximum value $y = -(b^2 - 4ac)/4a$ at $x = -b/2a$, then decreases in the interval $(-b/2a, +\infty)$; in this case it never attains a minimum value.

For example, we have for the quadratic function $y = x^2 - 5x + 4$: $a = 1 > 0$; the function thus has a minimum value. To find this, we do not need to draw the graph of the function; the above formulae show that the minimum value is obtained with $x = -b/2a = 2.5$ and is equal to $y = -(b^2 - 4ac)/4a = -2.25$.

It is worth mentioning that the maximum or minimum value of a quadratic function can be calculated by substituting $x = -b/2a$ in the expression for the function, instead of from the ready-made formula $y = -(b^2 - 4ac)/4a$.

We are faced with the task of finding the minimum or maximum values of a function fairly often, in very diverse problems. We shall consider two examples.

(1) Let us find the rectangle of maximum area from all the rectangles of given perimeter P .

On writing x for one side of the rectangle, we find that the other side is equal to $P/2 - x$. The area s of the rectangle is therefore

$$s = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2,$$

i.e. is a quadratic function of x . Since $a = -1 < 0$, the function has a maximum value; it is attained with $x = -b/2a = P/4$, and is equal to

$$\frac{P}{2} \cdot \frac{P}{4} - \left(\frac{P}{4}\right)^2 = \frac{P^2}{16}.$$

But the rectangle whose perimeter is P , and one of whose sides has a length $P/4$, is a square. We have thus shown that, of all the rectangles of a given perimeter, the square has the greatest area.

(2) Let us find how many seconds it takes a shell, shot vertically upwards from an anti-aircraft gun, to reach its maximum height above sea-level, and let us find this maximum height, air-resistance being neglected; we are given that the velocity of the shell as it leaves the gun is $v_0 = 500$ m/sec, the height of the gun above sea-level being $h_0 = 200$ m.

On neglecting the air-resistance, the motion of the shell will have constant deceleration $-g$. In this case, we know that the height h of the shell above sea-level at the instant t is

$$h = h_0 + v_0 t - \frac{g}{2} t^2.$$

Thus h is a quadratic function of t , where $a = -g/2 < 0$. The maximum value of function h (i.e. the maximum height to which the shell rises) will be obtained with

$$t = -\frac{b}{2a} = \frac{v_0}{g} = \frac{500}{9.8} \approx 51 \text{ sec.}$$

Thus the shell reaches its maximum height $h = H$ in 51 sec after the shot, whilst

$$H = 200 + 500 \cdot 51 - \frac{9.8}{2} \cdot 51^2 = 12955 \text{ m} \approx 13 \text{ km.}$$

The shell reaches a much smaller height in practice, due to air resistance.

The investigation of polynomials of the third and higher degrees becomes laborious and unwieldy with the elementary methods which we have used in studying polynomials of the first and second degrees. The task is facilitated by the methods of analysis.

19. Inverse proportionality and linear rational functions.

Definition. An inverse proportionality is a relationship defined by the formula

$$y = \frac{a}{x},$$

where a is a constant.

Its characteristic feature is that, when one variable changes, the other changes in the inverse ratio; in other words, if x_1 and y_1 , x_2 and y_2 are two pairs of corresponding values of x and y , and if $x_2 = kx_1$, then $y_2 = y_1/k$. For we have: $y_1 = a/x_1$, $y_2 = a/x_2$, but if $x_2 = kx_1$, then $y_2 = a/kx_1 = (a/x_1)/k = y_1/k$.

This property is expressed in words as: *y* is *inversely proportional* to *x*. The constant *a* is called the *coefficient of inverse proportionality*.

The formula for inverse proportionality can be rewritten in the form $xy = a$, in which the argument and the function (*x* and *y*) appear in complete symmetry. Hence if *y* is inversely proportional to *x*, *x* is in turn inversely proportional to *y*, with the same coefficient of inverse proportionality *a*.

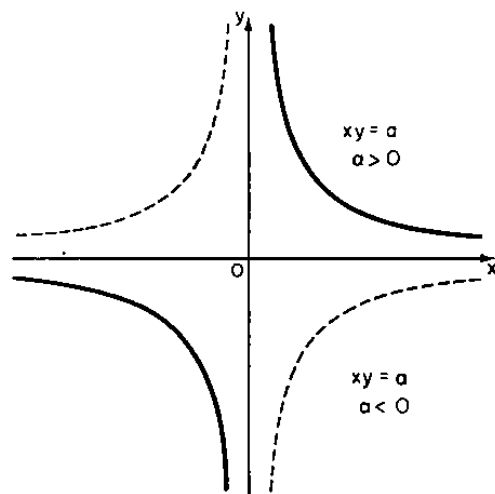


FIG. 14

As is shown in analytic geometry, the equation $xy = a$ is the equation of a rectangular hyperbola; its asymptotes are the co-ordinate axes, whilst its axes are equal to $2\sqrt{2|a|}$. Thus the graph of an inverse proportionality $y = a/x$ is a rectangular hyperbola; it is situated in the first and third quadrants if $a > 0$, and in the second and fourth if $a < 0$ (Fig. 14).

The function $y = a/x$ is defined throughout Ox except for the point $x = 0$. It is clear from the expression for the function that, when $|x|$ is very small, $|y|$ is very large, and conversely, when $|x|$ is very large, $|y|$ is very small.

If, therefore, say $a > 0$, when x increases through negative values, i.e. approaches zero from the left, the point of the graph will drop downwards whilst remaining in the third quadrant, whilst when x decreases through positive values, i.e. approaches zero from the right, the point of the graph will rise upwards whilst remaining in the first quadrant.

Inverse proportionality is a particular case of the relationship established by a linear rational function.

Definition. A **linear rational function** is a function of the form

$$y = \frac{ax + b}{cx + d}, \quad (*)$$

where a, b, c, d are constants.

This yields an inverse proportionality when $a = 0, d = 0$.

The linear rational function (*) is defined throughout Ox except for the point $x = -d/c$. The function is not defined at this point, since the denominator vanishes when $x = -d/c$.

With $c = 0$, the linear rational function becomes the linear function $y = (a/d)x + (b/d)$. We shall therefore assume that $c \neq 0$. We shall assume in addition that $bc - ad \neq 0$, since otherwise the function degenerates to a constant. For, let $bc = ad$; on writing m for the ratio a/c , we get: $a = cm, b = ad/c = dm$, so that

$$y = \frac{ax + b}{cx + d} = \frac{cmx + dm}{cx + d} = m \frac{cx + d}{cx + d} = m,$$

if $cx + d \neq 0$. It is clear from this that, with $bc - ad = 0$, the linear rational function coincides with the constant m throughout Ox , except for the point $x = -d/c$, where it is not defined.

It is easily shown by a parallel displacement of the system of co-ordinates that the graph of a linear rational function is a rectangular hyperbola, the asymptotes of which are parallel to the co-ordinate axes.

In fact, if we make a parallel displacement of the co-ordinate system, taking the new origin at the point $O'(-d/c, a/c)$, the equation of the graph of the linear rational function (*) in the system $O'x'y'$ becomes

$$y' = \frac{a'}{x'}, \quad \text{where} \quad a' = \frac{bc - ad}{c^2}$$

(we note that $bc - ad \neq 0$ by hypothesis, so that $a' \neq 0$). This is the equation of a rectangular hyperbola, whose asymptotes are the new co-ordinate axes, i.e. the straight lines

$$x = -\frac{d}{c}, \quad y = \frac{a}{c}.$$

Thus to draw the graph of the linear rational function, we have to draw through the point $(-d/c, a/c)$ straight lines parallel to the

co-ordinate axes, and, taking these as asymptotes, trace the rectangular hyperbola with axes equal to $2\sqrt{2|a|}$. Which quadrants, formed by the straight lines, are occupied by the hyperbola depends only on the sign of the coefficient $a' = (bc - ad)/c^2$, i.e. on the sign of $bc - ad$: if $bc - ad > 0$, the hyperbola is situated in the first and third quadrants, and if $bc - ad < 0$, it is in the second and fourth.

For example, let

$$y = \frac{x+1}{x-1}.$$

Here $a = 1$, $b = 1$, $c = 1$, $d = -1$, so that the asymptotes of the hyperbola are the straight lines $x = -d/c = 1$ and $y = a/c = 1$. Further, $a' = (bc - ad)/c^2 = 2$; consequently the axes of the hyperbola are equal to $2\sqrt{2|a'|} = 4$.

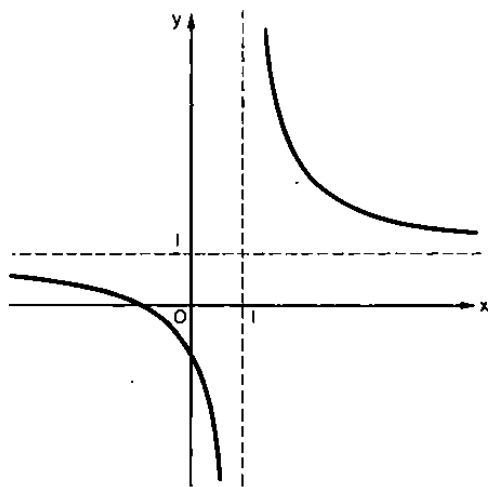


FIG. 15

The graph of this function is shown in Fig. 15; it is located in the first and third quadrants formed by the asymptotes, since here $bc - ad = 2 > 0$.

The course of variation of a linear rational function is easily seen from the graph. In fact:

The linear rational function () is either only increasing or only decreasing in any interval of Ox not containing the point $x = -d/c$.*

The nature of the variation (increase or decrease) depends on the sign of the expression $bc - ad$. If it is positive, the function is decreasing; if negative, the function is increasing.

Let us consider the behaviour of the function $y = (x + 1)/(x - 1)$ the graph of which is illustrated in Fig. 15. For values of x less than unity, the function always remains less than unity and decreases as x approaches unity; when the argument x passes from values less than unity to values greater than unity, the function makes an infinite jump, passing from negative values (as large as desired in absolute value) to arbitrarily large positive values; the function continues to decrease with further growth of x , whilst remaining greater than unity and approaching unity indefinitely as x increases indefinitely.

5. Inverse Functions. Power, Exponential and Logarithmic Functions

20. The concept of inverse function. Let y be given as a function of x :

$$y = f(x). \quad (\text{A})$$

We can regard y as the independent variable in this functional relationship; x then becomes a function of y :

$$x = \varphi(y). \quad (\text{B})$$

Notice that equations (A) and (B) express the same curve in the same system of axes Oxy , though the axis of the argument for the first function is Ox , whilst for the second it is Oy . The dependence of x on y , expressed by the function φ , will in general be different from the dependence of y on x , expressed by the function f .

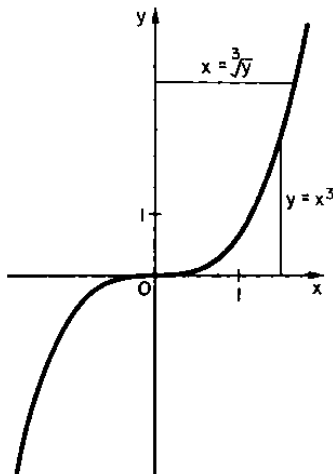


FIG. 16

For instance, if $y = x^3$, then $x = \sqrt[3]{y}$. The graph of these relationships is the so-called cubical parabola (Fig. 16). But whereas the first function is found from the argument by raising it to the power three, the second is found by taking the cube root of the argument. The nature of the dependence of the function on the argument is seen to be different.

The function φ is called the inverse of function f , and similarly f is called the inverse of function φ . For instance, taking the cube root defines a function which is the inverse of the function defined by raising to the power three (and vice versa).

Let y be given as a function of x by an equation solved for y . The inverse function, giving x as a function of y , is obtained by solving (if possible) this equation for x .

If we write x for the argument in formula (B), as in formula (A), and y for the function, we get

$$y = \varphi(x) \quad (C)$$

(in our example: $y = \sqrt[3]{x}$). It is usually in fact function (C) which is called the inverse of (A); in both the formulae, the same letter x denotes the argument, and the same letter y the function.

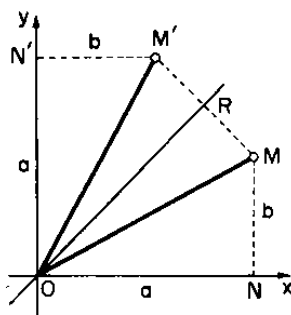


FIG. 17

The domain of definition of the inverse function is the set of values of the given function, whilst the set of values of the inverse function is the domain of definition of the given function.

There is a simple means of drawing the graph of the inverse function, based on the following property:

The graph of the inverse function $y = \varphi(x)$ is symmetrical to the graph of the direct function $y = f(x)$ with respect to the bisector of the first and third quadrants.

In fact, for $x = a$, let

$$y = f(a) = b.$$

The point $M(a, b)$ belongs to the graph of the direct function. The point M' with co-ordinate (b, a) now belongs to the graph of the inverse function, since $a = \varphi(b)$. We have (Fig. 17): $\triangle ON'M' = \triangle ONM$, in view of which $\angle N'OM' = \angle NOM$ and $OM' = OM$. Hence it follows, firstly that the bisector OR of the quadrant is in fact the bisector of the angle MOM' , and secondly, that $\triangle MOM'$ is isosceles. Therefore the bisector OR is the axis of symmetry of triangle MOM' . Thus to each point of the graph of the direct function there corresponds a point on the graph of the inverse, situated symmetrically to the first with respect to the bisector of the first and third quadrants. Since the converse is true, our assertion is fully proved.

Knowing the graph of a function, we can therefore find the graph of its inverse simply by rotating the figure about the bisector of the first and third quadrants.

Figure 18 illustrates the graphs of the linear function (I): $y = ax + b$, and its inverse (II): $y = \frac{x}{a} - \frac{b}{a}$ (also linear); Fig. 19 illustrates the graphs of (I): $y = x^3$ and its inverse (II): $y = \sqrt[3]{x}$.

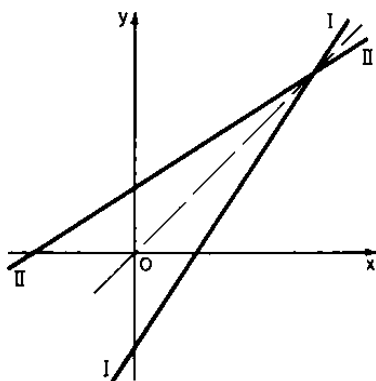


FIG. 18

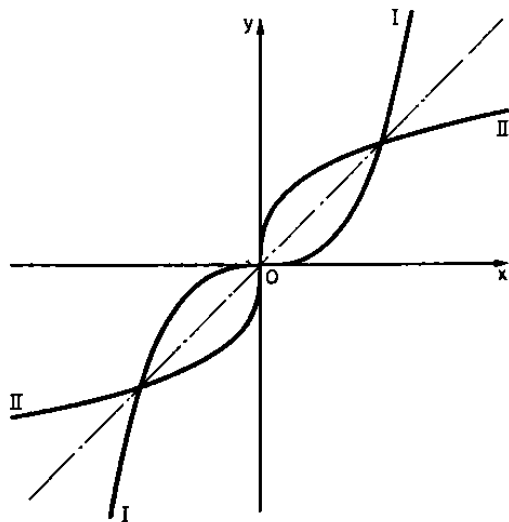


FIG. 19

It is important to notice that it does not at all follow, from the fact that a function is single-valued, that its inverse is single-valued. This is easily seen from two examples. Figure 20 (a) shows the graph

of a single-valued function; its inverse is not single-valued, because, for instance, to the value of the argument $y = OA$ there correspond at least three values of the function, represented by the segments AB , AC and AD . The function $y = x^2$ is single-valued; its inverse $y = \pm\sqrt{x}$ is two-valued.

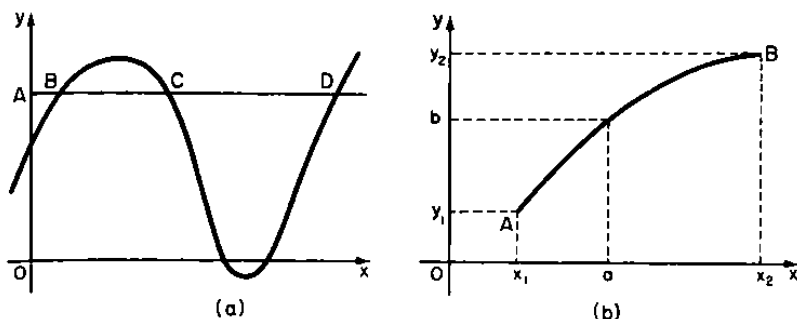


FIG. 20 (a) (b)

We shall indicate the condition in which the inverse of a single-valued function is also single-valued. This condition is monotonicity.

THEOREM. *If a single-valued function is monotonic in an interval (i.e. is either only increasing or only decreasing), its inverse is also single-valued and monotonic.*

Proof. If AB is the graph of the direct function, which is singlevalued and monotonic (say increasing) in the interval (x_1, x_2) (Fig. 20(b)), there cannot correspond more than one value $x = a$ of the inverse function to any value $y = b$ of the argument, since otherwise the direct function would have the same value $y = b$ for these values of x , i.e. it would not be monotonic as we assumed. It is obvious that the inverse is monotonic.

This is what we had to prove.

Notice that, if $f(x)$ is a single-valued and monotonic function and $\varphi(x)$ is its inverse, then

$$\varphi[f(x)] = x \quad \text{and} \quad f[\varphi(x)] = x.$$

This means that the successive application to any magnitude of the symbols of two mutually inverse and monotonic functions leaves this magnitude unchanged (such symbols so to speak cancel each other).

For example, $\sqrt[3]{x^3} = x$ and $(\sqrt[3]{x})^3 = x$.

21. Power functions. A power function *has the form*

$$y = x^n,$$

where the index n of the power is a real number.

As we shall see below, the domain of definiteness of the expression x^n is different for different n . But, whatever the n , the function $y = x^n$ is always defined for all positive values of x , i.e. in the open infinite interval $0 < x < \infty$.

As regards negative values of x and the value $x = 0$, the expression x^n may or may not be defined for these values, depending on the index n . Thus if n is a positive rational number, it can be written as the fraction p/q , where p and q are positive integers having no common factors. If the q is odd, $y = x^n$ is defined throughout Ox ; if p is even, (say for $n = 2, 4/3$) $y = x^n$ is an even function: its graph is symmetrical with respect to Oy ; whereas if p is odd (say for $n = 3, 5/3$), the function is odd: its graph is symmetrical with respect to the origin. In the case when q is even (which means that p is odd), the function $y = x^n$ is no longer defined for negative x . It is two-valued, its graph being symmetrical with respect to Ox ; finally, with n irrational and positive, the function is defined only for $x \geq 0$ and is single-valued. Similar circumstances hold with n negative, except for the fact that the function $y = x^n$ is now not defined at the point $x = 0$. It will thus be seen that, whatever the power n , we can find the entire graph of the power function $y = x^n$ simply by knowing the part of it that lies in the first quadrant.

The inverse of the power function $y = x^n$ is also a power function, with index equal to the reciprocal of the index of the direct function: $y = x^{1/n}$.

The behaviour of the power functions $y = x^n$ is essentially different with $n > 0$ and with $n < 0$.

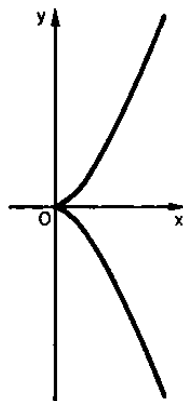
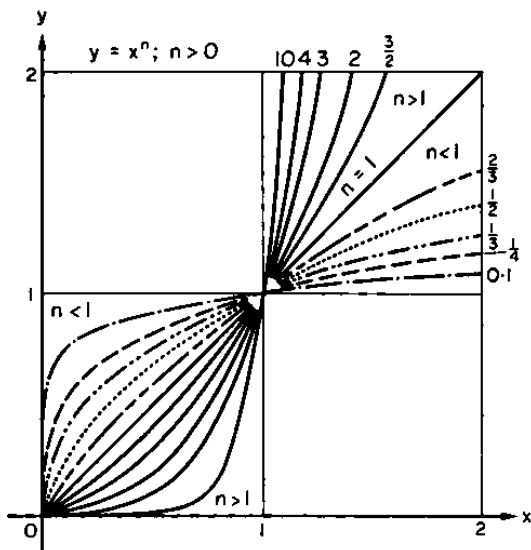


FIG. 22

FIG. 21

I. $n > 0$. Figure 21 illustrates the graphs of the functions $y = x^n$ in the first quadrant for different $n > 0$. Movable points rise upwards on all these curves: the functions $y = x^n$ are increasing in the interval $(0, \infty)$ for $n > 0$. Our curves pass through the points $(0, 0)$ and $(1, 1)$ and are split into two classes by the straight line $y = x$: those convex from below ($n > 1$) and those convex from above ($n < 1$).

The curves $y = x^n$ with $n > 0$ are called parabolas of the corresponding orders. Thus $y = x^2$ is a second order parabola, $y = x^3$ a third order parabola, or cubical parabola, $y = x^{\sqrt{2}}$ a parabola of order $\sqrt{2}$. A commonly encountered curve: $y = x^{3/2}$ (or $y^2 = x^3$) is called the semicubical parabola (Fig. 22).

It should be noted that every polynomial is a linear combination of power functions with positive integral indices.

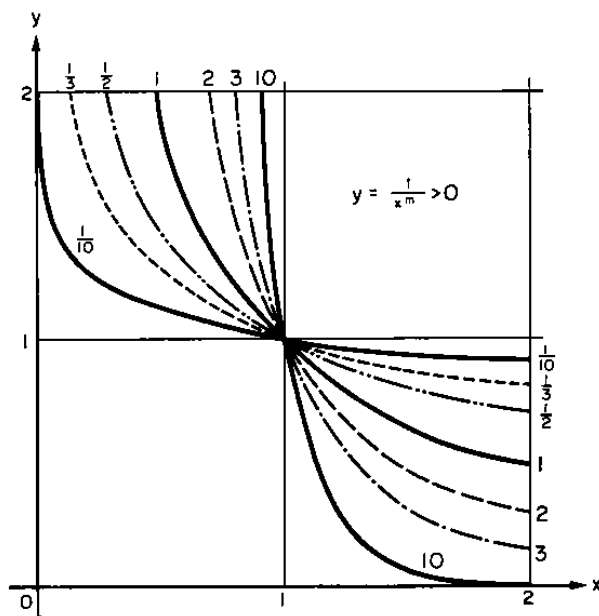


FIG. 23

II. $n < 0$. The graphs of $y = x^n$ for $n < 0$ are curves of an essentially different type to the parabolas above. Let $n = -m$, where $m > 0$. Figure 23 illustrates the graphs of the functions $y = 1/x^m$ for different $m > 0$ in the first quadrant. A point moving along any of these curves drops downwards: the function $y = x^n$ with $n < 0$ is decreasing in the interval $(0, \infty)$. When x increases indefinitely, y decreases and indefinitely approaches zero; conversely, when y in-

creases indefinitely, x decreases and indefinitely approaches zero. We conclude from this that, for any $m > 0$, the positive semi-axes Ox and Oy are asymptotes of the curve $y = 1/x^m$ (in the first quadrant). Our curves pass through the point $(1, 1)$ and are split by the rectangular hyperbola $y = 1/x$ into two classes: $m < 1$, $m > 1$.

The curves $y = 1/x^m$, $m > 0$, are called hyperbolas of the corresponding orders.

Many natural laws are expressed by means of power functions. For instance, we know from physics that the pressure of a gas is a power function of the volume (Poisson's law) when the volume changes adiabatically (i.e. without interchange of heat with the outside): $p = cv^{-\gamma}$, where c is a constant which is independent of the gas, and γ is a constant which depends on the number of atoms in a gas molecule ($\gamma > 1$). When the volume changes isothermally (i.e. at constant temperature), $\gamma = 1$, and we obtain the Boyle-Mariotte law: $p = c/v$.

The curves $y = x^n$ are sometimes called polytropic in physics and engineering.

22. Exponential and hyperbolic functions.

I. EXPONENTIAL FUNCTIONS. AN EXPONENTIAL FUNCTION HAS THE FORM

$$y = a^x.$$

When $a < 0$, the domain of definiteness of the expression a^x consists only of the rational numbers $x = p/q$, in which denominator q is an odd number. The exponential function is defined throughout Ox when the base is positive. Exponential functions are thus considered only on the assumption that the base is positive.

We note that $a^x > 0$ for any x and that $a^0 = 1$. The graphs of the exponential functions therefore lie above Ox for all (positive) bases a , and pass through the point $(0, 1)$. The behaviour of an exponential function depends essentially on whether $a > 1$ or $a < 1$ (with $a = 1$ the exponential function degenerates to a constant: $y = 1$).

If $a > 1$, y increases as the exponent x increases, and indefinite increase of the argument produces indefinite increase of the function. If $a < 1$, conversely, on indefinite increase of the argument the function decreases and approaches zero indefinitely.

The graph of an exponential function with base a is symmetrical about Oy with the graph of the exponential function with base $1/a$. For we can write the function $y = (1/a)^x$ as: $y = a^{-x}$, whence it is clear that it takes the same values for positive x as the function $y = a^x$ takes for negative x with the same absolute value, and vice

versa. This implies, in fact, that the graphs of $y = a^x$ and $y = (1/a)^x$ are symmetrical with respect to the axis of ordinates.

Since $1/a > 1$, if $0 < a < 1$, and conversely, $0 < 1/a < 1$ if $a > 1$, it will be seen that there corresponds to every graph of an exponential function with base less than unity the graph of an exponential function with base greater than unity, these graphs being symmetrical to each other with respect to the axis of ordinates.

Figure 24 illustrates the graphs of the exponential functions with $a = 2, 3, 10, 1/2, 1/3, 1/10$.

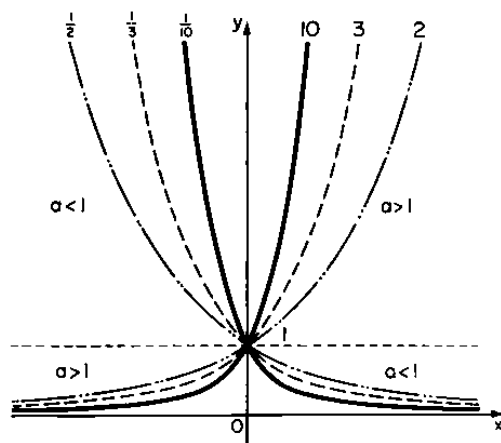


FIG. 24

The negative semi-axis Ox is an asymptote for the curves $y = a^x$ with $a > 1$, and the positive semi-axis Ox an asymptote for the curves $y = a^x$ with $a < 1$.

With $a > 1$, the exponential function represents a function which is "rapidly" increasing for large values of the independent variable, whilst its graph is a curve on which a moving point rises "steeply" upwards. Even a modest change in x (when x has large values) produces a comparatively large change in y . No matter what exponential function $y = a^x$ we take ($a > 1$), its growth finally outstrips that of the power function $y = x^n$ ($n > 0$) even when n is very large. In geometrical terms, every curve $y = a^x$ ($a > 1$), as from a certain point, has ordinates greater than the corresponding ordinates of the parabola $y = x^n$ ($n > 0$). This will be proved in Chapter IV.

The exponential functions are often encountered, in the most diverse problems. Here is an example.

Suppose a sum of A roubles is deposited in a bank, carrying p per cent compound interest per annum. This means that, after a year, p per cent is added to the initial sum, and in the following year the percentage which is added is

taken on the increased, and not on the original sum. It is readily seen that, after x years, the capital will be equal to

$$y = A \left(1 + \frac{p}{100} \right)^x$$

roubles.

In this example y is an exponential function of an integral argument x (multiplied by the constant coefficient A). Later (Chapter III) we shall consider the question of the "continuous" compound growth of a magnitude, leading to an exponential function whose argument varies continuously.

II. HYPERBOLIC* FUNCTIONS. Let us take the linear combination of exponential functions:

$$y = \frac{1}{2} a^x + \frac{1}{2} a^{-x} = \frac{a^x + a^{-x}}{2}, \quad a > 1.$$

The graph of the function y is obtained by addition of the corresponding ordinates of the graphs of the component functions $\frac{1}{2} a^x$ and $\frac{1}{2} a^{-x}$ (Fig. 25). We obtain a curve passing through the point $(0, 1)$ and symmetrical about Oy . In the first quadrant the curve indefinitely approaches the curve $y = \frac{1}{2} a^x$, and in the second the curve $y = \frac{1}{2} a^{-x}$. The behaviour of our function is evident from the graph. We shall only mention the fact that it is even, as follows from the expression for the function, and as is reflected in the symmetry of the graph.

The function

$$y = \frac{1}{2} a^x - \frac{1}{2} a^{-x} = \frac{a^x - a^{-x}}{2}, \quad a > 1,$$

is odd. Its graph (Fig. 26) passes through the origin and is symmetrical about it. In the first quadrant the curve is indefinitely close to the curve $y = \frac{1}{2} a^x$, and in the third to the curve $y = -\frac{1}{2} a^{-x}$.

We shall write $h_1(x)$ for the first of these functions:

$$h_1(x) = \frac{a^x + a^{-x}}{2},$$

and $h_2(x)$ for the second:

$$h_2(x) = \frac{a^x - a^{-x}}{2}.$$

Simple algebraic relationships can be established between them. Thus, it is easily shown that

$$h_1^2(x) - h_2^2(x) = 1, \quad h_1^2(x) + h_2^2(x) = h_1(2x)$$

* This name will be explained in Sec. 55.

for any x . These relationships recall the familiar relationships between the trigonometric functions $\cos x$ and $\sin x$. In general $h_1(x)$ has a number of properties resembling those of $\cos x$, whilst $h_2(x)$ is similar to $\sin x$. These functions, when the base has a definite value $a = e$, the approximate value of which is equal to 2.72 (see Sec. 41), are called: the first, $h_1(x)$, the hyperbolic cosine, written as $\cosh x$, or $\text{ch } x$, or $\cosh x$; the second, $h_2(x)$, the hyperbolic sine, written $\sinh x$, or $\text{sh } x$, or $\sinh x$. Use is also made of further hyperbolic functions, defined like the corresponding trigonometric functions, e.g. hyperbolic tangent, $\text{tg hyp } x$, $\text{th } x$, $\tanh x$:

$$\tanh x = \frac{\sinh x}{\cosh x}.$$

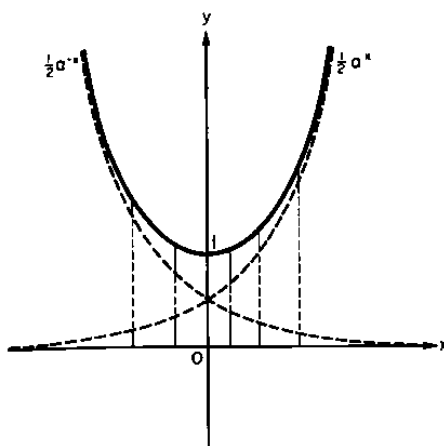


FIG. 25

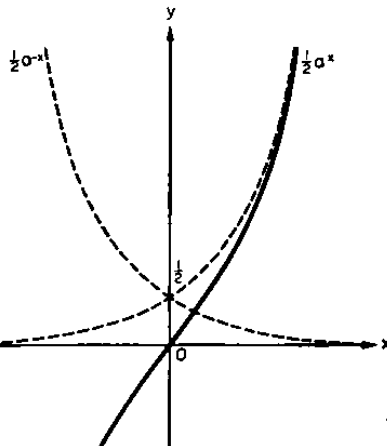


FIG. 26

23. Logarithmic functions. *The inverse of the exponential function $y = a^x$ is called the logarithmic function to base a :*

$$y = \log_a x.$$

Naturally, the base a is taken to be positive, as in the case of the exponential functions.

Taking into account the fact that the logarithmic and exponential functions are mutually inverse and monotonic, we have:

$$\log_a a^x = x \quad \text{and} \quad a^{\log_a x} = x.$$

We observe that, by the latter relationship, every power function $y = x^n$ with $x > 0$ and any index n can be written as a function of

a function, formed from exponential and logarithmic functions:

$$y = x^n = a^{n \log_a x}.$$

The graph of the logarithmic function (logarithmic) is easily drawn in accordance with the general rule for drawing the graph of an inverse function, starting from the graph of the corresponding exponential function $y = a^x$. On rotating Fig. 24 about the bisector of the first and third quadrants, we get the graph of the logarithmic functions with the same base a (Fig. 27).

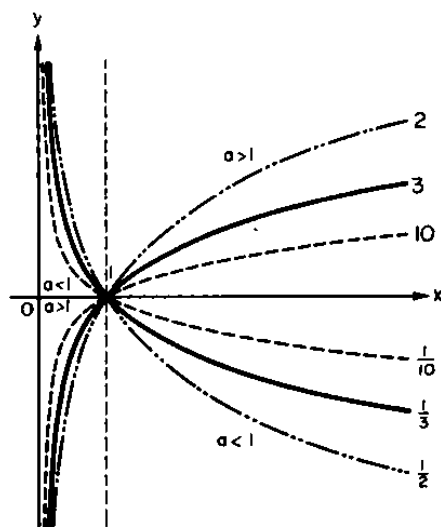


FIG. 27

Let us use these graphs to describe the behaviour of the logarithmic functions. We observe first of all that the logarithmic functions are defined throughout the positive semi-axis Ox and are not defined for negative or zero values of the independent variable. All the logarithmics pass through the point $(1, 0)$, i.e. the logarithm of unity is always equal to zero.

The behaviour of a logarithmic function depends essentially on whether $a > 1$ or $a < 1$. In the first case ($a > 1$) the logarithm is an increasing function throughout the interval $(0, \infty)$, being negative in the interval $(0, 1)$ and positive in the interval $(1, \infty)$. In the second case ($a < 1$) the logarithm is a decreasing function throughout the interval $(0, \infty)$, being positive in the interval $(0, 1)$ and negative in the interval $(1, \infty)$. If $a > 1$, the negative semi-axis of y is an asymp-

tote of the logarithmic, whilst if $a < 1$, the positive semi-axis of y is an asymptote.

The graph of the logarithmic function with base a is symmetrical about Ox with the graph of the logarithmic function with base $1/a$.

Let us take logarithmic functions with two different bases, a_1 and a_2 , i.e. $y_1 = \log_{a_1} x$ and $y_2 = \log_{a_2} x$. On expressing the x of the first equation in terms of y_1 and substituting in the second, we get: $y_2 = \log_{a_2} a_1^{y_1} = y_1 \log_{a_2} a_1$. We see that the logarithms of numbers to different bases are proportional to one another. To pass from one system of logarithms (with base a_1) to another system (with base a_2), we only need to multiply the logarithms of numbers in the first system by a constant ($\log_{a_2} a_1$). Consequently, one logarithmic is transformed into another by increasing or decreasing all its ordinates the same number of times.

When $a > 1$ the logarithm represents a function which increases "slowly" for large values of the independent variable, whilst the logarithmic is a curve along which a moving point rises "gradually" upwards. Variations of x (for large values of x) produce relatively small changes in the function. Unlike the case of the exponential function, any power function $y = x^n$ ($n > 0$) in the long run outstrips in its growth every logarithmic function $y = \log_a x$ ($a > 1$); in other words, the ordinates of the curve $y = x^n$, no matter how small the $n > 0$, will exceed, as from a certain point, the corresponding ordinates of the curve $y = \log_a x$ ($a > 1$). This will also be proved in Chapter IV.

6. Trigonometric and Inverse Trigonometric Functions

24. Trigonometric functions. *The trigonometric functions are*

$$y = \sin x, \quad y = \cos x, \quad y = \tan x,$$

together with the functions

$$y = \cot x, \quad y = \sec x, \quad y = \operatorname{cosec} x.$$

In mathematical analysis we always take the radian measure of the arc or angle as the independent variable x of the trigonometric functions. For instance, the value of the function $y = \sin x$ at $x = x_0$ is equal to the sine of the angle of x_0 radians (or roughly of $x_0 \cdot 57^\circ 17' 44.8''$). It is sometimes convenient to write the value of the argument in parts of π :

$$x_0 = \frac{x_0}{\pi} \cdot \pi \approx \frac{x_0}{3.14} \cdot \pi.$$

There exist between the six trigonometric functions: $y = \sin x$, $y = \cos x$, $y = \tan x$, $y = \operatorname{cosec} x$, $y = \sec x$, $y = \cot x$, five simple

algebraic relationships which are independent of each other and which are deduced from the definitions of the functions.

The trigonometric functions are periodic. Functions $\sin x$ and $\cos x$ (and hence also $\operatorname{cosec} x$ and $\sec x$) have period 2π , whilst functions $\tan x$ (and hence also $\cot x$) have period π .

We turn to the familiar graphs of the trigonometric functions. Let us start with $y = \sin x$. It is clear from its graph that this function increases from zero to unity in the interval $[0, \pi/2]$, decreases to -1 in the interval $[\pi/2, 3\pi/2]$, passing through zero at the point $x = \pi$, and finally, again increases to zero in the interval $[3\pi/2, 2\pi]$. Since the function $y = \sin x$ has period 2π , its entire graph (sine wave) is obtained by shifting the interval $[0, 2\pi]$ together with the corresponding part of the graph to the left and right by $2\pi, 4\pi, 6\pi, \dots$ (Fig. 28). As is easily seen from the graph, $y = \sin x$ is an odd function: it is symmetrical with respect to the origin.

The graph of the function $y = \cos x$ offers nothing essentially new, since

$$y = \cos x = \sin\left(x + \frac{\pi}{2}\right).$$

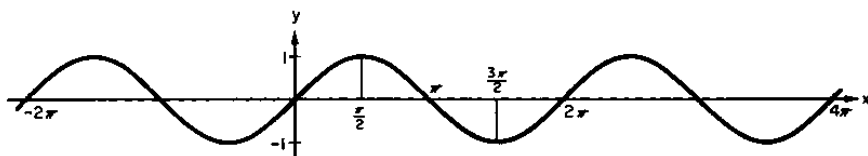


FIG. 28

This equation shows that, if the co-ordinate system is transformed in accordance with the formulae $x' = x + \pi/2$, $y' = y$, the equation of the curve takes the form $y' = \sin x'$ in the new system $O'x'y'$. The graph of $y = \cos x$ is therefore the sine wave discussed above, shifted to the left along Ox by $\pi/2$ (Fig. 29).

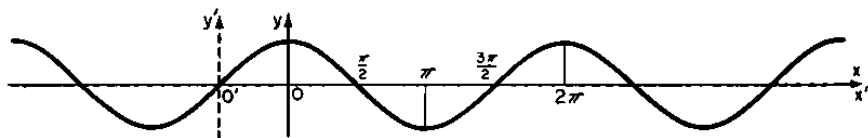


FIG. 29

The function $y = \cos x$ is decreasing in the interval $[0, \pi]$, from 1 to -1 , passing through 0 at the point $x = \pi/2$; then it increases

from -1 to 1 , in the interval $[\pi, 2\pi]$, passing through 0 at the point $x = 3\pi/2$. Both to the left of $x = 0$, and to the right of $x = 2\pi$, in the course of each interval of length 2π , starting from the point $2k\pi$, where k is any integer, the function $y = \cos x$ takes the same values, in the same sequence, as in the basic interval $[0, 2\pi]$. The function $y = \cos x$ is even, its graph being symmetrical about the axis of ordinates.

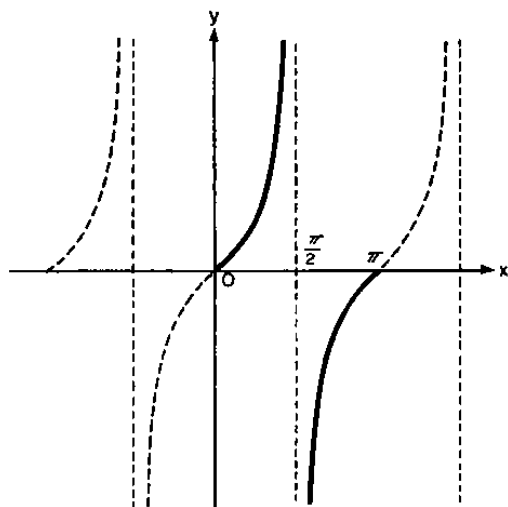


FIG. 30

The function $y = \tan x$ is defined throughout the interval $[0, \pi]$, except for the point $x = \pi/2$. When x indefinitely approaches $x = \pi/2$ from the left (increasing), y indefinitely increases through positive values; when x indefinitely approaches $x = \pi/2$ from the right (decreasing), y indefinitely increases in absolute value, whilst remaining negative. Since the function $y = \tan x$ has period π , the same picture is seen in the neighbourhood of any point $x = (2k + 1)\pi/2$, where k is any integer. The straight lines $x = (2k + 1)\pi/2$ are asymptotes of the graph of $y = \tan x$ (tangent curve). Figure 30 illustrates the tangent curve in the interval $[0, \pi]$ as a continuous line. The graph of $y = \tan x$ is obtained throughout Ox from the graph in the interval $[0, \pi]$ by simple repetition on the basis of the property of periodicity.

In every interval of definiteness of the function, $y = \tan x$ is an increasing function. Since $y = \tan x$ is an odd function, the graph is symmetrical about the origin.

We recommend the reader to draw the graphs of the three remaining trigonometric functions ($\operatorname{cosec} x$, $\sec x$, $\cot x$), then to describe their properties.

25. Simple and compound harmonic vibrations. The trigonometric functions have important applications in mathematics, natural science and engineering. They are encountered in almost all problems where we are concerned with periodic phenomena, i.e. phenomena which repeat themselves in the same sequence and in the same form after definite intervals of the argument (most commonly time).

The simplest of such phenomena are harmonic vibrations, in which a magnitude (e.g. the distance of the vibrating point from the equilibrium position) depends on time t in accordance with the law

$$y = A \sin(\omega t + \varphi_0).$$

This function is termed sinusoidal. The constant A^* is termed the amplitude of the vibration. It represents the maximum value which can be reached by y (during the vibrations). The argument $\omega t + \varphi_0$ of the sine is termed the phase of the vibration, whilst φ_0 , equal to the value of the phase at $t = 0$, is the initial phase. Finally, $\omega/2\pi$ (or sometimes ω itself) is termed the frequency of the vibration.

The origin of the last term will be clear from the connection between ω and the period T of the function (the period of the vibration). First of all, the function $y = A \sin(\omega t + \varphi_0)$ is periodic; its period $T = 2\pi/\omega$, since it is necessary to add $2\pi/\omega$ to the independent variable t to increase the phase by 2π . Therefore $\omega/2\pi = 1/T$, i.e. the number $\omega/2\pi$ shows how many periods are contained in unit time, i.e. how many times the periodic phenomenon is repeated in the course of unit time; this number thus in fact defines the frequency of the phenomenon. The number ω shows how many times the phenomenon is repeated in 2π units of time.

To draw the graph of the sinusoidal function

$$y = A \sin(\omega t + \varphi_0),$$

we only have to draw the sine wave in a new (auxiliary) co-ordinate system, obtained from the given system by scale transformations on the axes and by parallel displacement of the system along Ox (Sec. 16).

Figure 31 shows the graph of the function $y = 3 \sin(2t + \pi/3)$. The auxiliary co-ordinate system $O't''y'$ has been obtained in this example by halving the scale on Ot , tripling the scale on Oy , and shifting the system $\pi/3$ units of the new scale on Ot to the left.

* We can always assume A positive, since if it is negative, the function can be written as

$$y = -A \sin(\omega t + \varphi_0 + \pi).$$

The vibrations described by the equation $y = A \sin(\omega t + \varphi_0)$ are called *simple harmonic vibrations*, whilst their graphs are *simple harmonics*.

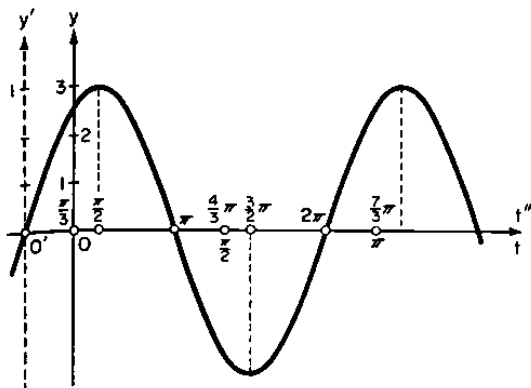


FIG. 31

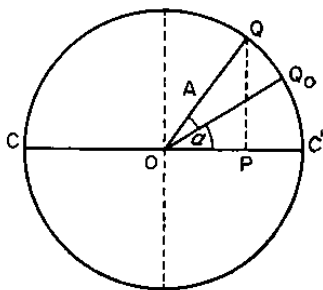


FIG. 32

We may take as an example of a simple harmonic vibration the motion of the horizontal projection of a point moving uniformly round a circle of radius A (Fig. 32). Let the initial position of the point be seen from the centre under an angle α (in radians) to the horizontal, and let the point, rotating anti-clockwise round the circle, go round in n times per second (n is not necessarily an integer).

In Fig. 32, Q_0 is the initial position of the point, Q is its position after t seconds from the start of the motion, P is the projection of Q on the horizontal.

How will the point P move, given the circular motion of point Q ? Let s be the distance of point P from the centre O of the circle, taken with the + or - sign depending on whether P is to the right or left of O . Then

$$s = OP = OQ \cos \angle QOP.$$

But

$$\angle QOP = \alpha + 2\pi nt, \quad OQ = A,$$

i.e.

$$s = A \cos(2\pi nt + \alpha) = A \sin\left(2\pi nt + \alpha + \frac{\pi}{2}\right),$$

or

$$s = A \sin(2\pi nt + \varphi_0), \quad \text{where} \quad \varphi_0 = \alpha + \frac{\pi}{2}.$$

We see that point P performs a harmonic vibration along the segment CC' . The amplitude of the vibration is equal to A , the period $T = 1/n$, the frequency is n , and the initial phase is $\alpha + \pi/2$. The projection of point Q on any other diameter will perform a harmonic vibration with the same amplitude and period, but with different initial phase.

We often encounter a sum of simple harmonic vibrations, i.e. a function of the form

$$s = A_1 \sin(\omega_1 t + \varphi_1) + A_2 \sin(\omega_2 t + \varphi_2) + \dots + A_n \sin(\omega_n t + \varphi_n),$$

where $A_1, A_2, \dots, A_n$ are constants; such a function is a linear combination of sinusoidal functions.

Suppose, for instance, that the point performing a harmonic vibration in accordance with the law $s = A_1 \sin(\omega_1 t + \varphi_1)$, is acted on further by a force which compels a fixed point to perform the vibration $s = A_2 \sin(\omega_2 t + \varphi_2)$. To find the relationship between the resultant deviation s of our point and time t , we find the "resultant" vibration, which must evidently be done by adding the "combined" vibrations.

In the general case the function

$$s = A_1 \sin(\omega_1 t + \varphi_1) + A_2 \sin(\omega_2 t + \varphi_2) \quad (\text{A})$$

will not be a simple harmonic vibration, i.e. will not be a function of the form $A \sin(\omega t + \varphi)$. But it will always be a periodic function if the ratio of the periods (or frequencies) is rational. In fact, let

$$T_1 : T_2 = \frac{2\pi}{\omega_1} : \frac{2\pi}{\omega_2} = \omega_2 : \omega_1 = p : q,$$

where p and q are integers having no common factors. The number $Z = T_1 q = T_2 p = 2\pi \omega / \omega_1 \omega_2$, where $\omega = \omega_1 p = \omega_2 q$, will now be the period of the total function.

If the frequencies are incommensurable, i.e. the ratio of the periods (or frequencies) is irrational, function (A) will not be periodic.

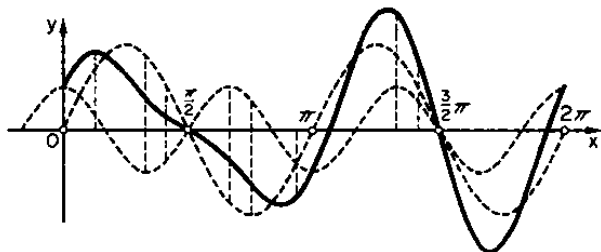


FIG. 33

The vibration obtained as a result of the addition of several simple harmonic vibrations is called compound, and its graph is a compound harmonic.

Figure 33 shows the graph of the function

$$y = 2 \sin 2x + \sin \left(3x + \frac{\pi}{2} \right)$$

(the graphs of the component functions are shown dotted). This is a periodic function with period 2π . Its graph is a compound harmonic, obtained as the result of "superimposing" two simple harmonic vibrations:

$$y = 2 \sin 2x \quad \text{and} \quad y = \sin \left(3x + \frac{\pi}{2} \right).$$

If the frequencies (i.e. the periods) of the component functions are the same: $\omega_1 = \omega_2 = \omega$, the sum will again be a sinusoidal function, and in this case there is no need to have recourse to addition of the graphs. We shall prove this. We have:

$$\begin{aligned} y &= A_1 \sin(\omega t + \varphi_1) + A_2 \sin(\omega t + \varphi_2) = \\ &= (A_1 \cos \varphi_1 + A_2 \cos \varphi_2) \cdot \sin \omega t + (A_1 \sin \varphi_1 + A_2 \sin \varphi_2) \cdot \cos \omega t = \\ &= A'_1 \sin \omega t + A'_2 \cos \omega t, \end{aligned}$$

where A'_1 and A'_2 denote the constant coefficients of $\sin \omega t$ and $\cos \omega t$. On multiplying and dividing the right-hand side by $A = \sqrt{A'^2_1 + A'^2_2}$, we get:

$$y = A \left(\frac{A'_1}{A} \sin \omega t + \frac{A'_2}{A} \cos \omega t \right).$$

Since $0 \leq A'_1/A \leq 1$, we can put $A'_1/A = \cos \varphi$, whence $A'_2/A = \sqrt{1 - (A'_1/A)^2} = \sin \varphi$.

We now get:

$$y = A (\cos \varphi \sin \omega t + \sin \varphi \cos \omega t) = A \sin(\omega t + \varphi),$$

which is what we wanted to prove.

Thus the "superimposition" of two simple harmonic vibrations with equal frequencies gives as a result a new simple harmonic vibration.

26. Inverse trigonometric functions. The functions

$$y = \text{Arcsin } x, \quad y = \text{Arccos } x, \quad y = \text{Arctan } x, \quad \text{etc.,}$$

inverse to the trigonometric functions $\sin x$, $\cos x$, $\tan x$ etc., are called the inverse trigonometric or inverse circular functions.

The values of these functions express radian measures of the arcs.

In view of the periodicity of the trigonometric functions, there exists an infinite set of arcs for which a given function has the same value. It follows from this that the corresponding inverse trigonometric function is infinitely-valued. We shall see, in fact, that if a straight line parallel to Ox cuts the graph of a trigonometric function, it cuts it in an infinite set of points, i.e. to a given value of the function there corresponds an infinity of values of the argument.

The graph of an inverse trigonometric function can be got by the general rule of drawing the curve symmetrical about the bisector of the first and third quadrants with the graph of the corresponding trigonometric function.

Thus the graph of the function $y = \text{Arcsin} x$ is a sine wave (Fig. 34). This function is defined only in the interval $-1 \leq x \leq 1$ (no arc exists whose sine has an absolute value exceeding 1). In this interval the function is infinitely-valued: every straight line, parallel to Oy and passing inside the interval $[-1, 1]$ cuts the curve $y = \text{Arcsin} x$ in an infinity of points. We shall distinguish the single-valued branch of this many-valued function by choosing the longest ("continuous") part of the graph which still cuts any line parallel to Oy in not more than one point. This can be done in various ways.

The part joining the points $(-1, -\pi/2)$ and $(1, \pi/2)$ is generally chosen (it is shown by a heavy line on the figure). Thus, from all the possible values of $\text{Arcsin} x$, corresponding to a given value of x , we distinguish the one which has an absolute value not exceeding $\pi/2$; it is written as $\arcsin x$ and is called the principal value of $\text{Arcsin} x$ (for the given x); thus

$$-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}.$$

$y = \arcsin x$ is a single-valued and increasing function, defined in the interval $[-1, 1]$.

We obtain the graph of the function $y = \text{Arccos} x$ if the graph of the function $y = \text{Arcsin} x$ is displaced $\pi/2$ downwards (Fig. 35). The function $y = \text{Arccos} x$ is also defined only in the interval $-1 \leq x \leq 1$ and is infinitely-valued. We distinguish from all the values of $y = \text{Arccos} x$ corresponding to a given x the one which lies between 0 and π ; it is termed the principal value of $\text{Arccos} x$ and is written as $\arccos x$:

$$0 \leq \arccos x \leq \pi.$$

The function $y = \arccos x$ is single-valued, positive and decreasing; it is defined in the interval $[-1, 1]$. Its graph is the part of the graph of $y = \text{Arccos} x$ shown as a heavy curve in Fig. 35.

The function $y = \text{Arctanz}$ is defined throughout the axis and is infinitely-valued. Its graph consists of an infinite set of "parallel" branches, lying in the strips

$$\dots, -\frac{3}{2}\pi \leq y \leq -\frac{\pi}{2}, \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \quad \frac{\pi}{2} \leq y \leq \frac{3}{2}\pi, \quad \dots$$

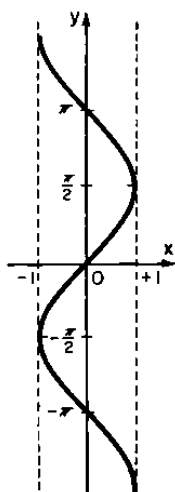


FIG. 34

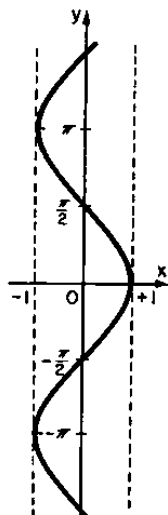


FIG. 35

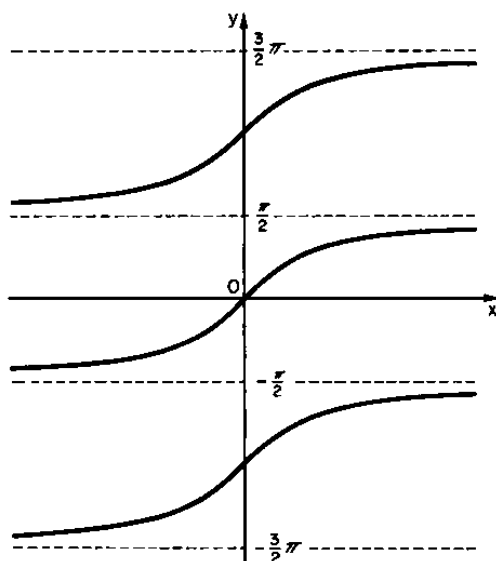


FIG. 36

(Fig. 36). Every straight line $y = (2k + 1)\pi/2$ (where k is any integer) is an asymptote of the corresponding branch of the function $y = \text{Arctan}x$. From all the values of this function, corresponding to a given x , we choose the one which lies between $-\pi/2$ and $\pi/2$; it is termed the principal value and is written as $\arctan x$:

$$-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}.$$

The function $y = \arctan x$ is single-valued, increasing, and defined for all x , $-\infty < x < \infty$. Its graph is distinguished by a heavy line in Fig. 36.

We recommend the reader to draw the graphs of the remaining three inverse trigonometric functions and to describe their properties.

Throughout what follows, unless there is a proviso to the contrary, we shall understand by inverse trigonometric functions their single-valued branched, formed by the principal values.

LIMITS

1. Basic Definitions

27. The limit of a function of an integral argument. Let $u_n = f(n)$ be a function of an integral argument, i.e. the sequence $u_1 = f(1)$, $u_2 = f(2)$, $\dots$, $u_n = f(n)$, $\dots$. It may happen that the value of u_n approaches a constant A when the argument n increases indefinitely, i.e. from a sufficiently large n , the difference between A and u_n becomes and remains smaller in absolute value than some previously assigned small number. We say in this case that A is the limit of the function $u_n = f(n)$.

Definition. The number A is called the *limit* of the function $u_n = f(n)$ or of the sequence $u_1, u_2, \dots, u_n, \dots$, if, given any positive number ϵ , however small, a value $n = N$ can be found such that, for all $n > N$,

$$|A - f(n)| < \epsilon. \quad (\dagger)$$

If A is the limit of $f(n)$, we also say that the function $f(n)$ or the sequence u_n tends to A as n tends to infinity. This is written as

$$\lim_{n \rightarrow \infty} f(n) = A, \quad \text{or} \quad f(n) \rightarrow A \quad \text{as} \quad n \rightarrow \infty$$

This is read as: the limit of $f(n)$ as n tends to infinity is equal to A .

The symbol $\lim$ is from the first three letters of the Latin word *limes* (limit), whilst $n \rightarrow \infty$ indicates that n increases indefinitely.

Let the function $f(n)$ have a limit equal to A as $n \rightarrow \infty$; then its values can be regarded as approximate values of the number A with an (absolute) error ϵ (see Sec. 8), which can be as small as desired. To find such an approximate value, we only need to take a sufficiently large n .

Let us take the example of the function $u_n = f(n) = n/(n + 1)$, i.e. the sequence

$$u_1 = \frac{1}{2}, \quad u_2 = \frac{2}{3}, \quad u_3 = \frac{3}{4}, \quad \dots, \quad u_n = \frac{n}{n+1}, \quad \dots$$

This sequence tends to unity. For, since

$$\left| 1 - \frac{n}{n+1} \right| = \frac{1}{n+1},$$

in order for $|1 - n/(n+1)|$ to be less than ε , where ε is any previously assigned positive number, we only need to satisfy the inequality

$$\frac{1}{n+1} < \varepsilon,$$

which is readily seen to hold for all $n > (1/\varepsilon) - 1$; thus for all n greater than the number $(1/\varepsilon) - 1$, the inequality(†) required by the definition of limit holds, i.e. $\lim_{n \rightarrow \infty} n/(n+1) = 1$. It will be observed

that, if we decrease ε , we increase the number of terms of the sequence such that all the following terms differ from unity by less than ε .

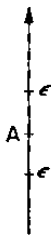


FIG. 37

Since only the absolute value of the difference between constant A and variable u_n appears in inequality (†), u_n can be both larger than or smaller than A , i.e. the function can approach its limit whilst increasing or decreasing. We turn to the geometrical interpretation: the values of function u_n will be represented by points of the real axis (Fig. 37). For the set of all values of u_n , there will be a corresponding set of points of the real axis, the order being known in which the variable u_n runs through the points of the set. We mark the point representing the number A — the limit of u_n as $n \rightarrow \infty$. By definition of the limit, the distance from the point A to the variable point u_n must be less than a previously assigned positive number ε , as from a certain position of u_n ; in other words, as from a certain position of the point u_n , it must move only inside an interval of length 2ε , the centre of which is the point A , i.e. inside an ε -neighbourhood of point A . Thus, speaking geometrically: *the point A is the limit of the variable point u_n if, from a certain n , the point u_n does not go*

outside a previously assigned ε -neighbourhood of the point A , however small.

Let $\varepsilon = \varepsilon_1$, $\varepsilon_1 > 0$; if $u_n \rightarrow A$, all the points u_n , starting from some $n = N_1$ (i.e. the point u_{N_1} and all the subsequent points: u_{N_1+1} , u_{N_1+2} ...) lie inside the ε_1 -neighbourhood of point A . We now decrease ε , letting $\varepsilon = \varepsilon_2$, where $0 < \varepsilon_2 < \varepsilon_1$; a certain number of our first points u_{N_1} , u_{N_1+1} , ... may now fall outside the new diminished neighbourhood of point A ; but, as from some $n = N_2$, where $N_2 > N_1$, all the points u_n (i.e. u_{N_2} , u_{N_2+1} , ...) will fall inside this ε_2 -neighbourhood of point A , and so on. As the argument n increases, the points u_n are so to speak compressed to a single point, viz. the point A .

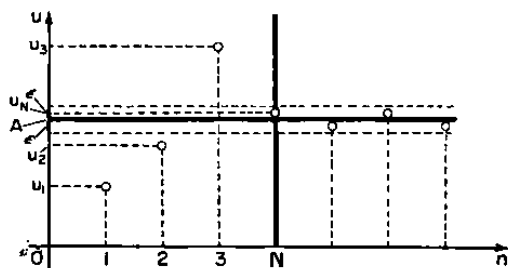


FIG. 38

If we take an axis for n in addition to the axis for u_n , and represent the function $u_n = f(n)$ geometrically by a graph in a system of Cartesian co-ordinates, the fact that the function has a limit will imply the following: whatever the ε , $\varepsilon > 0$, all the points of the graph situated to the right of some straight line $n = N$ will not go outside the strip bounded by the straight lines $u = A - \varepsilon$ and $u = A + \varepsilon$ (Fig. 38); the N , as from which this occurs, depends on the choice of ε , as already shown, and in general increases with decreasing ε (cf. our example).

If we regard the constant A as a function u_n , equal to A for all n , we can regard it as having a limit equal to itself: $\lim A = A$. For, the difference $A - A$ is equal to zero, i.e. less than any previously assigned positive number ε .

The definition of the limit of a function $f(n)$ does not of itself yield methods for finding the limit of a given function. We shall become acquainted with such methods later, and for the moment merely consider some examples illustrating the concept of limit.

28. Examples.

I. The sequence $1/2, 3/4, 7/8, 15/16, \dots, (2^n - 1)/2^n, \dots$ has a limit equal to 1. For

$$|1 - u_n| = \frac{1}{2^n},$$

and for $1/2^n$ to be less than a given positive ε , we only need to satisfy the inequality $2^n > 1/\varepsilon$, which follows from $n \log 2 > \log(1/\varepsilon)$ or $n > (\log 1/\varepsilon)/\log 2$; thus, given ε , we can always indicate an $n = N$ such that, for all $n > N$, we have

$$|1 - u_n| < \varepsilon,$$

which implies that 1 is the limit of the sequence.

The variable point u_n runs consecutively through the following points of the interval $[0, 1]$: u_1 is the mid-point of the interval, u_2 is the mid-point of the interval from u_1 to 1, u_3 is the mid-point of the interval from u_2 to 1 and so on. The points are compressed about the single point $u = 1$, the compression being from only one side, the left. In this example, u_n tends to its limit whilst increasing, i.e. it remains less than the limit.

It is also evident from this that the function $u_n = 1/2^n$ has a limit equal to 0; it tends to it whilst decreasing, i.e. whilst remaining greater than it. The points u_n are in this case compressed about $u = 0$ from the right.

II. We take the sequence

$$\sin \frac{\pi}{2}, \frac{1}{2} \sin 3 \frac{\pi}{2}, \frac{1}{3} \sin 5 \frac{\pi}{2}, \frac{1}{4} \sin 7 \frac{\pi}{2}, \dots, \frac{1}{n} \sin \left[(2n-1) \frac{\pi}{2} \right], \dots$$

The function $u_n = \frac{1}{n} \sin(2n-1) \pi/2$ tends to zero as $n \rightarrow \infty$, since $|0 - u_n| = 1/n < \varepsilon$ for all n satisfying the condition $n > 1/\varepsilon$. The difference $0 - u_n = -\frac{1}{n} \sin(2n-1) \pi/2 = (-1)^n/n$ will be positive or negative depending on whether n is even or odd. The values of u_n are alternately greater than and less than zero. On indefinitely approaching the value $u = 0$, the points lie alternately to the right and left of the zero point and are compressed round it. The variable tends to its limit whilst oscillating round it.

III. We recall a use of the concept of limit in elementary mathematics.

Let $a_1, a_2, \dots, a_n, \dots$ be an infinite geometrical progression with ratio q . We write s_n for the sum of the first n terms of this sequence:

$$\begin{aligned} s_n &= a_1 + a_2 + \dots + a_n = a_1(1 + q + \dots + q^{n-1}) = \\ &= a_1 \frac{1 - q^n}{1 - q} = \frac{a_1}{1 - q} - \frac{a_1}{1 - q} q^n. \end{aligned}$$

By increasing n in the sum, a larger number of terms of the progression is taken into account; thus we naturally regard *as the sum of the geometrical progression*: $s = a_1 + a_2 + \dots + a_n + \dots$ *the limit to which s_n tends as $n \rightarrow \infty$, if this limit exists.*

Let us verify that, if $|q| < 1$, the limit of s_n exists and is equal to $a_1/(1 - q)$. In fact:

$$\left| \frac{a_1}{1 - q} - a_1 \frac{1 - q^n}{1 - q} \right| = |a_1| \frac{|q|^n}{1 - q},$$

and simple working shows that, whatever the previously assigned positive number ε , the inequality

$$|a_1| \frac{|q|^n}{1 - q} < \varepsilon$$

is satisfied for all n satisfying the inequality

$$n > \frac{\log \left[(1 - q) \frac{\varepsilon}{|a_1|} \right]}{\log |q|}.$$

Notice, that if $0 < q < 1$, s_n tends to its limit either whilst increasing when $a_1 > 0$, or whilst decreasing, when $a_1 < 0$; whereas if $-1 < q < 0$, s_n tends to its limit whilst oscillating about it. It will be shown in Sec. 30 that s_n has no limit if $|q| \geq 1$.

IV. Some examples will be mentioned of sequences having no limits.

(1) We take the sequence

$$u_1 = \frac{1}{2}, \quad u_2 = \frac{1}{2}, \quad u_3 = \frac{3}{4}, \quad u_4 = \frac{1}{4}, \quad u_5 = \frac{7}{8}, \quad u_6 = \frac{1}{8}, \dots,$$

whose general term has the form $u_n = 1 - 1/2^{\frac{1}{2}(n+1)}$ if n is odd, and the form $u_n = 1/2^{\frac{1}{2}n}$ if n is even. We show that no number exists which is a limit of u_n . In fact, with a sufficiently large odd n , the corresponding value of u_n will be as close as desired to unity (see example I); but subsequent values of u_n do not all remain in this

situation; for even the next value u_{n+1} is as close to zero as u_n is to unity. The points successively occupied by the variable u_n are compressed about two points, 0 and 1, and not about a single point. As n varies, the point u_n jumps from a point close to 1 to a point close to 0. In view of this, u_n does not tend to a limit.

(2) The sequence

$$\sin \frac{\pi}{2}, \quad \sin 2 \frac{\pi}{2}, \quad \sin 3 \frac{\pi}{2}, \quad \dots, \quad \sin n \frac{\pi}{2}, \quad \dots$$

also has no limit, since $u_n = \sin n\pi/2$ for $n = 1, 2, 3, 4, \dots$ and takes consecutively the values 1, 0, -1, 0, then the same values again in the same order, and so on. There is no number which u_n approaches indefinitely.

If the terms of this sequence are taken with "convergence factors" (as those of sequence II), the new sequence $\sin \pi/2, \frac{1}{2} \sin 2\pi/2, \frac{1}{3} \sin 3\pi/2, \dots, \frac{1}{n} \sin n\pi/2, \dots$ will now have a limit, equal to zero.

The function $u_n = \frac{1}{n} \sin n\pi/2$ takes the value zero an unlimited number of times, i.e. the value of its limit. In other cases there may be no value of the argument for which the function takes the value of its limit.

(3) The function $u_n = 2n + 1$, the values of which, with $n = 1, 2, 3, \dots$ form the sequence of odd integers 1, 3, 5, $\dots$ does not tend to a limit, since the points u_n are compressed nowhere as $n \rightarrow \infty$; they spread away indefinitely.

29. The limit of a function of a continuous argument. We turn to the concept of limit for a function $y = f(x)$ of the continuous argument x .

I. The limit as $x \rightarrow x_0$. Let the independent variable x tend to (approach indefinitely) a number x_0 , this being written as: $x \rightarrow x_0$. This means that we assign to x values differing by as little as desired from (but not equal to) x_0 . As regards the corresponding values of $f(x)$, it may happen that they approach some number A . This is taken to mean that, for all x sufficiently close to x_0 , the absolute value of the difference between A and $f(x)$ is less than any previously assigned small positive number. In such a case we say that A is the limit of the function $y = f(x)$ as $x \rightarrow x_0$.

Definition. We call A the limit of the function $y = f(x)$ as $x \rightarrow x_0$ if, given any positive ε , however small, a positive number δ can be found such that, for all x differing from x_0 and satisfying the inequality

$|x_0 - x| < \delta$, we have

$$|A - f(x)| < \varepsilon.$$

This is written as

$$\lim_{x \rightarrow x_0} f(x) = A \quad \text{or} \quad f(x) \rightarrow A, \quad x \rightarrow x_0$$

and we say alternatively that $f(x)$ tends to A as $x \rightarrow x_0$.

We can also say that:

$y = f(x)$ tends to A as x tends to x_0 if, for all points x ($x \neq x_0$) contained in some sufficiently small δ -neighbourhood of the point $x = x_0$, the values of the function $y = f(x)$ are contained in as small an ε -neighbourhood as desired of the point $y = A$.

If $x \rightarrow x_0$, we shall call x_0 the "limit" point for x .

Attention must be paid to the fact that our definition of limit does not require that $f(x)$ be specified at the limit point x_0 ; we only require $f(x)$ to be defined for all x in any neighbourhood of the point x_0 , excluding, possibly, the point x_0 itself. For instance, the function $y = (x - 3)/(x^2 - 9)$ is not defined at $x = 3$, but it has a limit equal to $1/6$ as $x \rightarrow 3$ (see example 3 of p. 85). Similarly the function $y = \frac{\sin x}{x}$ is not defined for $x = 0$, but, as will be shown in Sec. 40, it tends to 1 as x tends to zero.

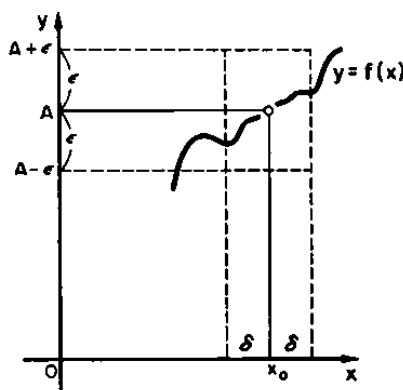


FIG. 39

The existence of a limit of $f(x)$ equal to A as $x \rightarrow x_0$ is geometrically illustrated thus (Fig. 39): we erect perpendiculars on Ox at x_0 and on Oy at A to their intersection, and arbitrarily assign a positive number ε ; a δ -neighbourhood of the point x_0 is then found such that

all the points of the graph of $f(x)$ corresponding to points $x (x \neq x_0)$ lying in this neighbourhood will lie in the strip bounded by the straight lines $y = A - \varepsilon$ and $y = A + \varepsilon$.

The number δ depends on the choice of ε ; in general it decreases as ε decreases.

We sometimes have occasion to consider problems about the limit of a function $f(x)$ when x cannot take all values sufficiently close to x_0 (as stated in our definition), but only takes values larger than x_0 , which is expressed in words as: " x tends to x_0 from the right". Similarly, we sometimes consider values of x sufficiently close to x_0 but always less than x_0 , which is expressed in words as: " x tends to x_0 from the left". Finally, values of x may be considered equal only to the terms of some sequence which tends to x_0 . In these cases the limit of a function is defined as above, except that we do not indicate in the statement all values of x sufficiently close to x_0 , but only values greater than x_0 , or less than x_0 , or belonging to a given sequence that tends to x_0 . To distinguish these particular methods whereby x tends to x_0 from the general method of our original definition, it is sometimes said in the latter case that x tends arbitrarily to x_0 .

If the limit of $f(x)$ exists as x tends to x_0 from the left, it is called the "left-hand limit" of $f(x)$ at x_0 ; similarly for the "right-hand limit". The left and right-hand limits are written respectively as $f(x_0 - 0)$, $f(x_0 + 0)$. Thus:

$$\lim_{\substack{x \rightarrow x_0 \\ x < x_0}} f(x) = f(x_0 - 0), \quad \lim_{\substack{x \rightarrow x_0 \\ x > x_0}} f(x) = f(x_0 + 0).$$

Notice that, if $f(x) \rightarrow A$ when x tends arbitrarily to x_0 , the same will be true for any particular method of approach of x to x_0 (either only from the right, or only from the left, and so on). The converse also holds: if $f(x) \rightarrow A$ as $x \rightarrow x_0$ by any special method, then $f(x) \rightarrow A$ when $x \rightarrow x_0$ arbitrarily. We shall not dwell on the proof of this proposition.

Examples. (1) Let us show that $\lim_{x \rightarrow 1} (2x - 1) = 1$. We assign a positive ε . To obtain $|1 - (2x - 1)| < \varepsilon$ or $2|1 - x| < \varepsilon$, we need to satisfy the inequality

$$|1 - x| < \frac{\varepsilon}{2};$$

we can therefore take $\delta = \frac{1}{2}\varepsilon$. We see that, given any $\varepsilon > 0$, the values of the function $y = 2x - 1$ differ from 1 by not more than ε in a

$\frac{1}{2}\varepsilon$ -neighbourhood of the point $x = 1$; hence by definition, 1 is in fact the limit of the function when x tends to 1 arbitrarily. Now let $x \rightarrow 1$ only through values of some sequence, say $1/2, 3/4, \dots, (2^n - 1)/2^n, \dots$. Now, $u_n = 2(2^n - 1)/2^n - 1$ also tends to 1 as $n \rightarrow \infty$. For, in accordance with Sec. 27,

$$\left| 1 - \left(2 \frac{2^n - 1}{2^n} - 1 \right) \right| = 2 \cdot \left(1 - \frac{2^n - 1}{2^n} \right) = \frac{1}{2^{n-1}}$$

will be less than ε ($\varepsilon > 0$) for all n greater than $\frac{\log 1/\varepsilon}{\log 2} + 1$.

(2) Let us show that

$$\lim_{x \rightarrow -1} \frac{2x + 1}{3 + x} = -\frac{1}{2}.$$

We want to show that, given any positive number ε , a positive δ can be found such that, with $|x + 1| < \delta$, we have

$$\left| -\frac{1}{2} - \frac{2x + 1}{3 + x} \right| = \left| \frac{1}{2} + \frac{2x + 1}{3 + x} \right| < \varepsilon.$$

This inequality is readily transformed to

$$\left| \frac{x + 1}{3 + x} \right| < \frac{2}{5} \varepsilon \quad \text{or} \quad \left| \frac{2}{1 + x} + 1 \right| > \frac{5}{2\varepsilon}.$$

Since the absolute value of a sum is not less than the difference of the absolute values of its terms: $|a + b| \geq |a| - |b|$, the last inequality is satisfied if

$$\left| \frac{2}{1 + x} \right| - 1 > \frac{5}{2\varepsilon} \quad \text{or} \quad \left| \frac{1}{1 + x} \right| > \frac{5 + 2\varepsilon}{4\varepsilon},$$

i.e.

$$|1 + x| < \frac{4\varepsilon}{5 + 2\varepsilon}.$$

Thus our original inequality is satisfied for all x satisfying this final inequality, i.e. the number $\delta = 4\varepsilon/(5 + 2\varepsilon)$ possesses the required property.

(3) Let us show that $\lim_{x \rightarrow 3} \frac{x - 3}{x^2 - 9} = 1/6$. Here, the function is not defined at the limit point.

Let ε be an arbitrary positive number. We have to prove the existence of a δ -neighbourhood of the point $x = 3$ such that, for all points x of it (excluding $x = 3$), the inequality is satisfied:

$$\left| \frac{1}{6} - \frac{x - 3}{x^2 - 9} \right| < \varepsilon.$$

This inequality reduces to the following:

$$\left| \frac{x^2 - 6x + 9}{x^2 - 9} \right| = \left| \frac{x - 3}{x + 3} \right| < 6\varepsilon.$$

We were justified in cancelling the fraction by $x - 3$, since $x \neq 3$, i.e. $x - 3$ does not vanish.

We have further:

$$\left| \frac{x + 3}{x - 3} \right| = \left| 1 + \frac{6}{x - 3} \right| > \frac{1}{6\varepsilon},$$

and this inequality will hold for all x satisfying

$$\frac{6}{|x - 3|} - 1 > \frac{1}{6\varepsilon}, \quad \text{i.e.} \quad \frac{6}{|x - 3|} > \frac{1 + 6\varepsilon}{6\varepsilon},$$

whence

$$|x - 3| < \frac{36\varepsilon}{1 + 6\varepsilon}.$$

We have thus proved the existence of the required δ -neighbourhood of the point $x = 3$; we can take $36\varepsilon/(1 + 6\varepsilon)$ as δ .

II. The limit as $x \rightarrow \infty$. Now let the independent variable x increase indefinitely (in absolute value), or as we can also say, "tend to infinity: $x \rightarrow \infty$ ". This simply implies the following: x takes any values greater (in absolute value) than any desired positive number*. It may happen that, as $|x|$ increases, the values of $f(x)$ approach a number A ; A is then called the limit of $f(x)$ as $x \rightarrow \infty$, and we say of $f(x)$ that it tends to A as $x \rightarrow \infty$.

Definition. We call A the limit of the function $y = f(x)$ as $x \rightarrow \infty$ if, given any small positive number ε , a positive number N can be found such that, for all x satisfying $|x| > N$, we have

$$|A - f(x)| < \varepsilon.$$

This is written as:

$$\lim_{x \rightarrow \infty} f(x) = A \quad \text{or} \quad f(x) \rightarrow A \quad \text{as} \quad x \rightarrow \infty.$$

The definition implies geometrically that:

For all points x lying outside a sufficiently large N -neighbourhood of the point $x = 0$, the values of the function $y = f(x)$ are contained in as small an ε -neighbourhood as desired of the point $y = A$ (Fig. 40).

* We naturally assume that the function $f(x)$ is defined for all these values of x .

If x can in fact take any values exceeding in absolute value some positive number, we sometimes say for the sake of accuracy that " x tends arbitrarily to infinity", instead of simply " x tends to infinity". But other cases are encountered, in which x tends to infinity in particular ways. For instance, x can increase indefinitely in absolute value whilst preserving its sign: positive, which is written as: $x \rightarrow +\infty$ (in words: " x tends to positive infinity"), or negative, which is written as: $x \rightarrow -\infty$ (in words: " x tends to negative infinity"). In addition, x can tend to ∞ (or to $+\infty$ or to $-\infty$) "along a sequence", i.e. only taking values equal to the terms of a sequence which tends to ∞ (or to $+\infty$ or to $-\infty$).

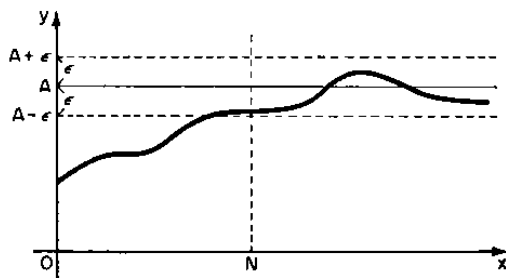


FIG. 40

The reader will be able to introduce for himself the changes in the definition of the limit of a function when the argument tends to infinity in these special ways. The definition for the case of an integral argument (x takes only values 1, 2, 3, ...) with which we began our study of limits (Sec. 27) follows at once from our general definition. Obviously, if $f(x) \rightarrow A$ as x tends arbitrarily to infinity, $f(x)$ will have the same limit when x tends to infinity in some special way. The converse also holds (we omit the proof).

Examples. (1) We take the exponential function $y = a^x$. If $a > 1$, then

$$\lim_{x \rightarrow -\infty} a^x = 0.$$

We have, in fact:

$$a^x < \varepsilon$$

for $x < (\log \varepsilon)/(\log a)$, where ε is a previously assigned positive number, as small as desired. The fraction $\frac{\log \varepsilon}{\log a}$ will be seen to be negative, since the denominator is positive and the numerator negative by virtue of the natural assumption that $\varepsilon < 1$. If $a < 1$,

then

$$\lim_{x \rightarrow +\infty} a^x = 0,$$

since $a^x < \varepsilon$ for $x > \frac{\log \varepsilon}{\log a}$ *. Notice that the fraction $\frac{\log \varepsilon}{\log a}$ is positive here, since the denominator and numerator are both negative.

These limit equations for the exponential functions can clearly be seen from the graphs (see Fig. 24).

(2) The function $y = \frac{x+1}{x-1}$ tends to 1 if x tends to infinity in an arbitrary manner:

$$\lim_{x \rightarrow \infty} \frac{x+1}{x-1} = 1.$$

For, to satisfy the inequality $\left| 1 - \frac{x+1}{x-1} \right| = |2/(x-1)| < \varepsilon$, i.e. $|x-1| > 2/\varepsilon$, where ε is a previously assigned positive number, as small as desired, it is sufficient to have $|x| - 1 > 2/\varepsilon$ or $|x| > \frac{2}{\varepsilon} + 1$. Thus we have $\left| 1 - \frac{x+1}{x-1} \right| < \varepsilon$ for all x satisfying the condition $|x| > N = \frac{2}{\varepsilon} + 1$. This is what we had to prove.

We also see from the graph of the function (Fig. 15) that, no matter how far removed the point x towards infinity, the ordinate approaches unity indefinitely.

2. Non-Finite Magnitudes. Rules for Passage to the Limit

30. Infinitely large magnitudes. Bounded functions.

I. INFINITELY LARGE MAGNITUDES. Let us consider the "behaviour in the limit" of the function $y = f(x)$, i.e. its variation as $x \rightarrow x_0$ (or $x \rightarrow \infty$), when it increases indefinitely in absolute value; we then term $y = f(x)$ an "infinitely large magnitude".

Definition. The function $y = f(x)$ is called an *infinitely large magnitude* as $x \rightarrow x_0$ (or as $x \rightarrow \infty$) if, given any large positive num-

* As in the first case, to find the values of x for which $a^x < \varepsilon$, we take logarithms: $x \log a < \log \varepsilon$; on dividing both terms of this inequality by $\log a$, which is positive for $a > 1$ and negative for $a < 1$, we obtain an inequality for x in the same sense in the first case, and in the opposite sense in the second case.

ber M , a positive number δ (or N) can be found such that, for all x satisfying the condition $0 < |x_0 - x| < \delta^*$ (or the condition $|x| > N$), we have

$$|f(x)| > M.$$

An infinitely large magnitude $y = f(x)$ is characterized geometrically by the fact that the moving point representing it lies outside an arbitrarily large M -neighbourhood of the point $y = 0$ for values of x sufficiently close to x_0 as $x \rightarrow x_0$ (Fig. 41) (or for values of x sufficiently large, as $x \rightarrow \infty$ (Fig. 42)).

A function $y = f(x)$ which is an infinitely large magnitude as $x \rightarrow x_0$ (or $x \rightarrow \infty$) has no limit in the ordinary sense. But the point representing it becomes indefinitely remote from $y = 0$ along Oy as $x \rightarrow x_0$ (or $x \rightarrow \infty$); with the aim of reflecting this regularity in its "behaviour in the limit" in words and writing, we say that " $f(x)$ tends to infinity" and write:

$$\lim_{\substack{x \rightarrow x_0 \\ (x \rightarrow \infty)}} f(x) = \infty \quad \text{or} \quad f(x) \rightarrow \infty.$$

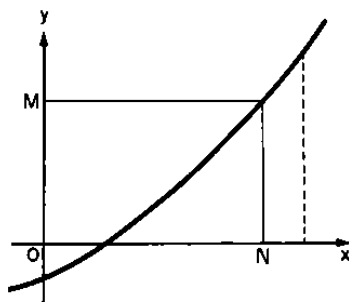
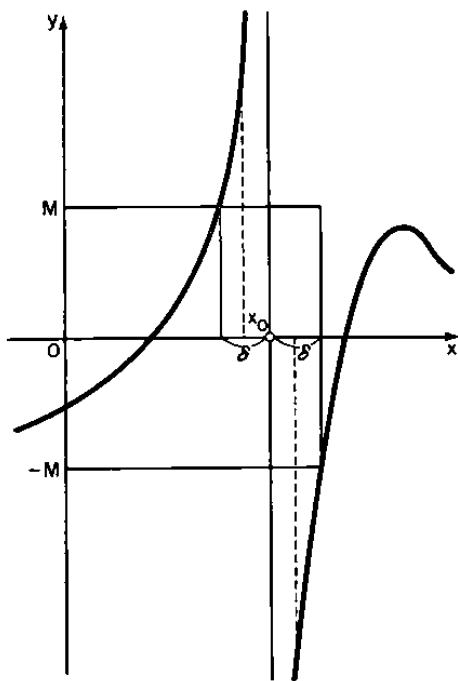


FIG. 42

FIG. 41

* The left-hand inequality indicates that the value $x = x_0$ is not taken.

According to our definition, an infinitely large magnitude $f(x)$ can take both positive and negative values. For instance, the function of an integral argument $f(n) = (-2)^n$, i.e. the sequence $-2, 4, -8, 16, -32, \dots, (-2)^n, \dots$ is an infinitely large magnitude as $n \rightarrow \infty$.

Suppose that the infinitely large magnitude $y = f(x)$ takes either only positive or only negative values in a neighbourhood of the point $x = x_0$ as $x \rightarrow x_0$ (or outside a neighbourhood of the point $x = 0$ as $x \rightarrow \infty$). This means that, as x approaches x_0 (or as x moves away from 0), the point $y = f(x)$ moves away indefinitely from $y = 0$ along Oy only on one side: upwards, or downwards. We express this feature in the behaviour of $f(x)$ in the limit as follows: " $f(x)$ tends to positive (or negative) infinity", which is written as:

$$\lim_{\substack{x \rightarrow x_0 \\ (x \rightarrow \infty)}} f(x) = +\infty, \quad \lim_{\substack{x \rightarrow x_0 \\ (x \rightarrow \infty)}} f(x) = -\infty.$$

If we are not concerned with the sign of an infinitely large magnitude or if it does not retain one sign, we keep our previous expression (simply "infinity"; the symbol: ∞).

One should always bear in mind the conventionality of these expressions and notations, and remember that infinity (∞) is not a number, so that there is no sense whatever in speaking of operations on ∞ .

Every number, no matter how large, is finite. A very large number must not be confused with an infinitely large magnitude.

The definition that we have given of an infinitely large magnitude refers to the case when x tends arbitrarily to x_0 (or to ∞). The definition is very easily modified to relate to the case when x tends to x_0 (or to ∞) in a particular way.

Examples. (1) We show that the sum s_n of the first n terms of a geometrical progression (see example III, Sec. 28) is an infinitely large magnitude as $n \rightarrow \infty$ if the absolute value of the ratio of the progression is greater than unity.

Let M be a previously assigned positive number. For the absolute value of s_n to exceed M , we have to have

$$|a_1| |q^n - 1| > M |q - 1|,$$

where q is the ratio of the progression, whilst a_1 is the first term. This inequality will certainly be satisfied if n is chosen so that

$$|a_1| (|q|^n - 1) > M |q - 1|,$$

since $|q^n - 1| \geq |q|^n - 1$. Thus

$$|q|^n > 1 + |q - 1| \frac{M}{|a_1|} \quad \text{and} \quad n > \frac{\log\left(1 + |q - 1| \frac{M}{|a_1|}\right)}{\log|q|} = N$$

(since $|q| > 1$, we have $\log|q| > 0$ and the inequality can be divided by $\log|q|$).

Thus $|s_n| > M$ for all $n > N$. This is what we had to prove.

The same holds good if $q = 1$, since now $s_n = a_1 \cdot n$. But if $q = -1$, $s_n = a_1$ for n odd and $s_n = 0$ for n even. As the sum s_n varies it takes the values a_1 and 0 in turn, and thus has no limit.

We have therefore proved our statement of Sec. 28.

(2) The function $y = 1/x$ tends to infinity as $x \rightarrow 0$. The nature of the unbounded increase of the function here depends on how x tends to zero. If $x \rightarrow 0$ from the right, $1/x$ is a positive infinitely large magnitude: $\lim 1/x = +\infty$; if $x \rightarrow 0$ from the left, $1/x$ is a negative infinitely large magnitude: $\lim 1/x = -\infty$; if $x \rightarrow 0$ taking both positive and negative values in any neighbourhood of the point $x = 0$, $1/x$ is simply an infinitely large magnitude: $\lim 1/x = \infty$. All these circumstances are clearly reflected in the graph of the function (the rectangular hyperbola whose asymptotes are the co-ordinate axes (see Fig. 14)).

(3) The function $y = (x + 1)/(x - 1)$ has a similar behaviour, as $x \rightarrow 1$, as the function $y = 1/x$ as $x \rightarrow 0$ (see Sec. 19). In fact:

$\lim_{x \rightarrow 1} \frac{x + 1}{x - 1} = \infty$. We have $\lim_{x \rightarrow 1} \frac{x + 1}{x - 1} = +\infty$ if x tends to 1 from the right, and $\lim_{x \rightarrow 1} \frac{x + 1}{x - 1} = -\infty$ if x tends to 1 from the left (see Fig. 15 for the graph of the function $y = \frac{x + 1}{x - 1}$).

(4) As $x \rightarrow 0$ the function $y = 1/x^2$ is a positive infinitely large magnitude. For, in order to have $1/x^2 > M$, we simply need $x^2 < 1/M$, i.e. $|x| < +\frac{1}{\sqrt{M}}$. Thus for all x for which $|0 - x| < \delta = +\frac{1}{\sqrt{M}}$, the value of $y = x^2$ is greater than any previously assigned positive M . Thus $\lim_{x \rightarrow 0} 1/x^2 = +\infty$.

(5) $\lim_{x \rightarrow \frac{1}{2}\pi} \tan x$ is equal to $+\infty$ or $-\infty$ depending on whether x tends to $\frac{1}{2}\pi$ from the left or right.

(6) The reader will readily show that the exponential function $y = a^x$ is a positive infinitely large magnitude as $x \rightarrow +\infty$ if $a > 1$, and as $x \rightarrow -\infty$ if $a < 1$.

II. BOUNDED FUNCTIONS.

Definition. A function $y = f(x)$ is said to be *bounded* in some domain of variation of the argument x if there exists a positive number M such that, for all x belonging to this domain, we have $|f(x)| \leq M$ (or what amounts to the same thing, $-M \leq f(x) \leq M$). If there is no such number M , the function is said to be *unbounded*.

The point y representing a bounded function $y = f(x)$ does not go outside the interval $[-M, M]$ on Oy .

Definition. A function is said to be *bounded as $x \rightarrow x_0$* (or as $x \rightarrow \infty$) if the function is bounded in some neighbourhood of the point x_0 (or outside some neighbourhood of the point 0).

Obviously, every constant is bounded.

If $f(x)$ is an unbounded function, there is at least one sequence $x_1, x_2, \dots, x_n, \dots$ for which $f(x)$ increases indefinitely in absolute value, i.e. is an infinitely large magnitude:

$$\lim_{n \rightarrow \infty} f(x_n) = \infty.$$

Every infinitely large magnitude is an unbounded function; but the converse is false: an unbounded function may not be an infinitely large magnitude. Thus the function $y = x \sin x$ as $x \rightarrow \infty$ (or what amounts to the same thing, $y = \frac{1}{x} \sin \frac{1}{x}$ as $x \rightarrow 0$) is an unbounded function. For we cannot find an M such that, for all x , $|x \sin x| \leq M$; whatever M we take, if x runs through e.g. the sequence $x_1 = \pi/2$, $x_2 = 3\pi/2$, $x_3 = 5\pi/2$, $\dots$, $x_n = (2n-1)\pi/2$, $\dots$, the

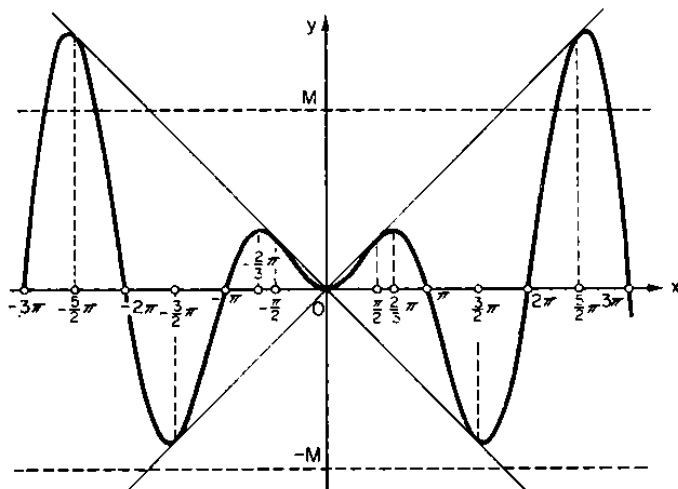


FIG. 43

corresponding values of $|x \sin x|$ are $\pi/2, 3\pi/2, 5\pi/2, \dots, (2n-1)\pi/2, \dots$, and become in the end (i.e. for sufficiently large n) greater than our number M . On the other hand, we can easily indicate a sequence which also tends to infinity, say $x_1 = \pi, x_2 = 2\pi, x_3 = 3\pi, \dots, x_n = n\pi, \dots$, for which the function always remains zero, i.e. has a limit ($= 0$). Therefore $y = x \sin x$, whilst being an unbounded function, is not an infinitely large magnitude (as $x \rightarrow \infty$). It is clear from the graph of $y = x \sin x$ (Fig. 43) that, for sufficiently large x , there are points of the graph which lie outside the strip $-M \leq y \leq M$, where M is a positive number as large as desired; but the points of the graph corresponding to further growth of $|x|$ do not remain in this situation (as must be the case for an infinitely large magnitude), but drop back within the strip and even reach Ox .

31. Infinitesimals. A case of special interest arises when the function tends to zero for some given variation of the argument. By the definition of limit, a function $f(x)$ tends to zero as $x \rightarrow x_0$ (or as $x \rightarrow \infty$) if, given a positive number ε , however small, a corresponding δ (or N) can be found such that, for all x satisfying $0 < |x_0 - x| < \delta$ (or $|x| > N$), we have $|f(x)| < \varepsilon$.

Definition. A function $f(x)$, tending to zero as $x \rightarrow x_0$ (or as $x \rightarrow \infty$), is called an **infinitesimal as $x \rightarrow x_0$ or in the neighbourhood of the point x_0 (or at infinity).**

We may mention as examples of infinitesimals: $y = x - 1$ as $x \rightarrow 1$; $y = a^x$ as $x \rightarrow +\infty$ if $a < 1$, and as $x \rightarrow -\infty$ if $a > 1$, etc.

Every number, no matter how small, is finite. A very small number should never be confused with an infinitesimal.

Zero is the only number which may be regarded as an infinitesimal (by virtue of the fact that the limit of a constant is equal to itself).

An infinitesimal is a bounded magnitude.

There is a simple connection between infinitely large magnitudes and infinitesimals.

THEOREM. If the function $f(x)$ is an infinitely large magnitude, $1/f(x)$ is an infinitesimal; if $f(x)$ is an infinitesimal, $1/f(x)$ is an infinitely large magnitude.

Proof. Let $f(x) \rightarrow \infty$ as $x \rightarrow x_0$ (or as $x \rightarrow \infty$); we have to show that $1/f(x) \rightarrow 0$.

We assign an arbitrary $\varepsilon > 0$ and take $M = 1/\varepsilon$; there corresponds to this M a δ (or an N) such that, for all x satisfying the condition $0 < |x_0 - x| < \delta$ (or $|x| > N$), we have

$$|f(x)| > M = \frac{1}{\varepsilon},$$

whence

$$\left| \frac{1}{f(x)} \right| < \frac{1}{M} = \varepsilon,$$

which implies that $\lim_{x \rightarrow x_0} 1/f(x) = 0$.

The proof of the second part of the theorem is similar. But it is essential to assume here that $f(x)$ does not vanish for any values of x in some neighbourhood of the point $x = x_0$ if $x \rightarrow x_0$ (or outside some neighbourhood of $x = 0$ if $x \rightarrow \infty$), since otherwise the fraction $1/f(x)$ becomes meaningless in any neighbourhood of $x = x_0$ (or outside any neighbourhood of $x = 0$), so that there will be no possibility of considering the question of the limit of the fraction.

The following theorems will be very useful in what follows.

THEOREM. A function which has a limit can be written as the sum of a constant, equal to the limit, and an infinitesimal.

Proof. Let $\lim_{x \rightarrow x_0} f(x) = A$. Then $|f(x) - A| < \varepsilon$ for all x sufficiently close to x_0 (or sufficiently large), the ε being arbitrary; and this implies that $f(x) - A$ is an infinitesimal, by definition. Consequently,

$$f(x) - A = \alpha(x) \quad \text{or} \quad f(x) = A + \alpha(x),$$

where $\alpha(x)$ is an infinitesimal as $x \rightarrow x_0$ (or as $x \rightarrow \infty$).

CONVERSE. If a function can be written as the sum of a constant and an infinitesimal, the constant is the limit of the function.

Proof. It follows from the equation $f(x) = A + \alpha(x)$, where A is a constant and $\alpha(x)$ is an infinitesimal as $x \rightarrow x_0$ (or as $x \rightarrow \infty$) that

$$|f(x) - A| = |\alpha(x)| < \varepsilon$$

for all x sufficiently close to x_0 (or sufficiently large), with an arbitrary positive ε . But this means that $f(x)$ has the limit A for the variation of x in question:

$$\lim_{x \rightarrow x_0} f(x) = A.$$

This is what we had to prove.

All the arguments in this section assume that x tends to x_0 (or to ∞) arbitrarily. It is perfectly obvious how the statements should be modified if x tends to x_0 (or to ∞) in some special way.

32. Rules for passage to the limit. We have so far only been able to verify, on the basis of the definition of limit, whether a given number is the limit of a given function for a given variation of its argument. There are rules, however, with the aid of which it is often possible to find directly the limits of functions (rules for passage to the limit).

Before turning to the simplest of these rules, we shall consider some theorems on infinitesimals, and then, as corollaries, some theorems on functions which tend to a limit, not necessarily zero.

For the sake of brevity we shall sometimes speak of a function u (or v, w, α, β etc.) tending to a limit, and write $\lim u$, without indicating precisely how the independent variable varies. However, when speaking of several functions, we shall always assume that they tend to their limits with the same variation of the independent variables.

THEOREM I. The sum (in the algebraic sense) of two, three and in general any finite number of infinitesimals is an infinitesimal.

Proof. Let $\alpha, \beta, \dots, \eta$ be k infinitesimals (k is a definite number). We show that their sum

$$\omega = \alpha + \beta + \dots + \eta$$

is also an infinitesimal. We assign an arbitrarily small positive number ε and take the number ε/k . Let x be the argument of all these infinitesimals. We now have $|\alpha| < \varepsilon/k$ in some δ -neighbourhood of the point $x = x_0$, if $x \rightarrow x_0$ (or outside some N -neighbourhood of $x = 0$ if $x \rightarrow \infty$). Similarly, we have in the corresponding neighbourhoods: $|\beta| < \varepsilon/k, \dots, |\eta| < \varepsilon/k$. Since $|\alpha| = |\alpha + \beta + \dots + \eta| \leq |\alpha| + |\beta| + \dots + |\eta|$, we have in the least of these δ -neighbourhoods of x_0 , if $x \rightarrow x_0$ (or in the greatest of the N -neighbourhoods of zero, if $x \rightarrow \infty$):

$$|\omega| < \underbrace{\frac{\varepsilon}{k} + \frac{\varepsilon}{k} + \dots + \frac{\varepsilon}{k}}_{k \text{ terms}} = k \cdot \frac{\varepsilon}{k} = \varepsilon.$$

Thus, there exists, for every positive ε , a neighbourhood of $x = x_0$ (or a neighbourhood of $x = 0$) such that $|\omega| < \varepsilon$ in it (or outside it). This proves that ω is an infinitesimal.

The special proviso regarding the definiteness of the number of terms is essential. The point is that we have occasion to consider in analysis special sums in which both the terms themselves and the number of them vary. The theorem is not concerned with such sums.

If, for instance, we let the number of terms increase indefinitely whilst each tends to zero, the assertion of the theorem may be false. Let

$$\alpha = \frac{1}{n^2}, \quad \beta = \frac{2}{n^2}, \quad \dots, \quad \eta = \frac{n}{n^2};$$

as $n \rightarrow \infty$ each of these magnitudes tends to zero, but at the same time the number of them increases. The sum

$$\omega = \frac{1 + 2 + \dots + n}{n^2} = \frac{n(n+1)}{2n^2} = \frac{1}{2} + \frac{1}{2n},$$

in fact tends to $\frac{1}{2}$ as $n \rightarrow \infty$ and is by no means an infinitesimal.

The following is an immediate corollary of theorem I.

THEOREM I'. The limit of the sum of two, three and in general any finite number of terms, is equal to the sum of the limits of these terms.

Proof. Let $u, v, \dots, t$ be k terms (k is a definite number) tending respectively to the limits $a, b, \dots, d$. Let

$$w = u + v + \dots + t.$$

We shall prove that

$$\begin{aligned} \lim w &= \lim(u + v + \dots + t) = \lim u + \lim v + \dots + \lim t \\ &= a + b + \dots + d. \end{aligned}$$

We have:

$$u = a + \alpha, \quad v = b + \beta, \quad \dots, \quad t = d + \tau,$$

where $\alpha, \beta, \dots, \tau$ are infinitesimals. Consequently,

$$\begin{aligned} w &= (a + b + \dots + d) + (\alpha + \beta + \dots + \tau) \\ &= (a + b + \dots + d) + \omega, \end{aligned}$$

where $\omega = \alpha + \beta + \dots + \tau$, being the sum of k infinitesimals, is an infinitesimal by theorem I. Since w is equal to the sum of an infinitesimal ω and a constant $a + b + \dots + d$, the latter is in fact the limit of w .

THEOREM II. The product of a bounded function and an infinitesimal is an infinitesimal.

Proof. Let α be an infinitesimal as $x \rightarrow x_0$ (or as $x \rightarrow \infty$), and let the function u be bounded, $|u| \leq M$ in some neighbourhood of the point x_0 , if $x \rightarrow x_0$ (or outside some neighbourhood of $x = 0$, if $x \rightarrow \infty$). We shall prove that $\lim(u\alpha) = 0$.

We assign an arbitrarily small positive number ε . In some neighbourhood of $x = x_0$ (or outside some neighbourhood of $x = 0$), α satisfies the inequality $|\alpha| < \varepsilon/M$; hence

$$|u\alpha| = |u| |\alpha| < M \cdot \frac{\varepsilon}{M} = \varepsilon,$$

whence the theorem follows.

In particular,

$$\lim(c\alpha) = 0,$$

if c is a constant, and $\alpha \rightarrow 0$.

The assertion of the theorem may be false if u is not bounded (e.g.: $n \cdot (1/n) = 1$).

Since an infinitesimal is bounded, the product of two infinitesimals is an infinitesimal.

Theorems I and II lead to:

THEOREM II'. The limit of the product of two, three and in general a finite number of factors is equal to the product of the limits of these factors.

Proof. Retaining the notation of theorem I', we prove that

$$\lim w = \lim(uv \dots t) = \lim u \cdot \lim v \dots \lim t = ab \dots d.$$

We first take the product of two factors: u and v . We have:

$$u = a + \alpha, \quad v = b + \beta$$

i.e.

$$w = uv = (a + \alpha)(b + \beta) = ab + (\alpha b + \beta a + \alpha\beta).$$

The quantity in brackets is an infinitesimal by theorems II and I, so that

$$\lim w = ab.$$

The theorem is now easily extended to any finite number of factors. Suppose we have three factors: u, v, t . Then

$$\begin{aligned} \lim w &= \lim(uvt) = \lim[(uv) \cdot t] = \lim(uv) \cdot \lim t \\ &= \lim u \cdot \lim v \cdot \lim t. \end{aligned}$$

The theorem is similarly proved in the case of four factors then five, and so on.

It follows, in particular, from this theorem that

(1) a constant factor can be taken outside the limit sign:

$$\lim cu = c \lim u$$

(since the limit of a constant is equal to the constant) and

(2) the limit of a positive integral power of a function is equal to the same power of the limit of the function:

$$\lim u^n = \lim (uu \dots u) = \lim u \cdot \lim u \dots \lim u = (\lim u)^n.$$

THEOREM III. When an infinitesimal is divided by a function whose limit is non-zero, the resulting quotient is an infinitesimal.

Proof. Let α be an infinitesimal and u a function whose limit differs from zero: $\lim_{\substack{x \rightarrow x_0 \\ (x \rightarrow \infty)}} u = a \neq 0$. The theorem says that $\frac{\alpha}{u} \rightarrow 0$.

We show first of all that $1/u$ is a bounded function. Say we assign an $\varepsilon = \frac{1}{2}|a|$; then there exists a δ -neighbourhood of the point $x = x_0$, if $x \rightarrow x_0$ (or an N -neighbourhood of $x = 0$, if $x \rightarrow \infty$) such that in it (or outside it):

$$|a - u| < \frac{|a|}{2};$$

since $|a - u| > |a| - |u|$, we have

$$|a| - |u| < \frac{|a|}{2},$$

or

$$|u| > \frac{|a|}{2},$$

whence

$$\left| \frac{1}{u} \right| < \frac{2}{|a|},$$

which in fact shows that $1/u$ is a bounded function as $x \rightarrow x_0$ (or as $x \rightarrow \infty$).

Since the quotient α/u can be regarded as the product of the bounded function $1/u$ and the infinitesimal α , we have by theorem II:

$$\lim \frac{\alpha}{u} = 0.$$

This is what we had to prove.

The result of dividing an infinitesimal by an infinitesimal may be either an infinitesimal or an infinitely large magnitude; or it may

have a non-zero limit, or no limit at all, either finite or infinite. We shall see examples of this later.

THEOREM III'. The limit of a quotient is equal to the quotient of the limits, provided the denominator does not tend to zero.

Proof. Let $\lim u = a$ and $\lim v = b \neq 0$, so that $u = a + \alpha$, $v = b + \beta$, where α and β are infinitesimals. We prove that

$$\lim w = \lim \frac{u}{v} = \frac{\lim u}{\lim v} = \frac{a}{b}.$$

In fact,

$$w = \frac{u}{v} = \frac{a + \alpha}{b + \beta} = \frac{a}{b} + \frac{b\alpha - a\beta}{b(b + \beta)},$$

the fraction $\frac{b\alpha - a\beta}{b(b + \beta)}$ being an infinitesimal by theorem III: its numerator is an infinitesimal by the previous theorems, whilst the limit of its denominator is equal to b^2 , which is non-zero by hypothesis. Thus $\lim w = a/b$. If $b = 0$, the theorem becomes meaningless.

The above theorems yield simple rules for finding the limits of functions. These limits are got by carrying out arithmetical operations on functions whose limits are already known.

33. Examples. We shall solve the following examples by using the rules indicated.

(1) Find the limit of $u_n = (2^n - 1)/2^n$ as $n \rightarrow \infty$ (Sec. 28, I).

We have:

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{2^n}\right) = \lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{1}{2^n} = 1, \quad \text{since } \lim_{n \rightarrow \infty} \frac{1}{2^n} = 0.$$

(2) Find the limit of $s_n = a_1(1 - q^n)/(1 - q)$ as $n \rightarrow \infty$ (Sec. 28, III).

We have:

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} a_1 \frac{1 - q^n}{1 - q} = \lim_{n \rightarrow \infty} \frac{a_1}{1 - q} - \frac{a_1}{1 - q} \lim_{n \rightarrow \infty} q^n,$$

but $\lim q^n$ is equal to zero or infinity as $n \rightarrow \infty$, depending on whether $|q| < 1$ or > 1 . In the first case we get $\lim s_n = a_1/(1 - q)$, in the second $\lim s_n = \infty$.

(3) Find the limit of $y = 2x - 1$ as $x \rightarrow 1$ (Sec. 29, 1).

We have:

$$\lim_{x \rightarrow 1} (2x - 1) = \lim_{x \rightarrow 1} (2x) - \lim_{x \rightarrow 1} 1 = 2 \lim_{x \rightarrow 1} x - 1 = 2 \cdot 1 - 1 = 1.$$

(4) Find the limit of $y = (2x + 1)/(3 + x)$ as $x \rightarrow -1$ (Sec. 29, 2).

We have:

$$\lim_{x \rightarrow -1} \frac{2x + 1}{3 + x} = \frac{\lim(2x + 1)}{\lim(3 + x)} = \frac{2 \lim x + 1}{3 + \lim x} = \frac{2 \cdot (-1) + 1}{3 - 1} = -\frac{1}{2}.$$

In examples (3) and (4) it has proved sufficient to replace x by its limit in the expression for the function in order to find the limit of the function. We shall see in art. 3 that *this method holds good for any elementary function at all points of its domain of definition*. We shall prove the method here only for linear rational functions, where the limit of the argument is not a zero of the denominator.

(5) Find the limit of $y = (a_0 x^m + a_1 x^{m-1} + \dots + a_m)/(b_0 x^n + b_1 x^{n-1} + \dots + b_n)$ as $x \rightarrow x_0$, if we know that $b_0 x_0^n + b_1 x_0^{n-1} + \dots + b_n \neq 0$.

We have:

$$\begin{aligned} \lim_{x \rightarrow x_0} y &= \frac{\lim(a_0 x^m + a_1 x^{m-1} + \dots + a_m)}{\lim(b_0 x^n + b_1 x^{n-1} + \dots + b_n)} \\ &= \frac{a_0(\lim x)^m + a_1(\lim x)^{m-1} + \dots + a_m}{b_0(\lim x)^n + b_1(\lim x)^{n-1} + \dots + b_n} \\ &= \frac{a_0 x_0^m + a_1 x_0^{m-1} + \dots + a_m}{b_0 x_0^n + b_1 x_0^{n-1} + \dots + b_n}. \end{aligned}$$

For example,

$$\lim_{x \rightarrow 2} \frac{3x^3 - 2x^2 + x + 1}{x^2 - 5x + 3} = \frac{3 \cdot 2^3 - 2 \cdot 2^2 + 2 + 1}{2^2 - 5 \cdot 2 + 3} = -6\frac{1}{3}.$$

If the denominator of the linear rational function vanishes at $x = x_0$, the theorem on the limit of a quotient becomes meaningless and a special discussion is needed.

(6) Find $\lim_{x \rightarrow 3} (x - 3)/(x^2 - 9)$ (Sec. 29, 3).

When $x \rightarrow 3$, the denominator tends to zero, but the numerator also tends to zero. We have the case of a quotient of two infinitesimals.

But since $x^2 - 9 = (x - 3)(x + 3)$, $\frac{x - 3}{x^2 - 9} = 1/(x + 3)$ for all $x \neq 3$.

We thus have, by the definition of limit:

$$\lim_{x \rightarrow 3} \frac{x - 3}{x^2 - 9} = \lim_{x \rightarrow 3} \frac{x - 3}{(x - 3)(x + 3)} = \lim_{x \rightarrow 3} \frac{1}{x + 3} = \frac{1}{6}.$$

(7) Find $\lim_{x \rightarrow 1} \frac{2x - 3}{x^2 - 5x + 4}$.

The limit here is ∞ , since the denominator is an infinitesimal as $x \rightarrow 1$, whilst the numerator does not tend to zero. In fact, $\lim_{x \rightarrow 1} (x^2 - 5x + 4) = 1^2 - 5 \cdot 1 + 4 = 0$, whilst $\lim_{x \rightarrow 1} (2x - 3) = 2 \cdot 1 - 3 = -1$; hence by theorem III:

$$\lim_{x \rightarrow 1} \frac{x^2 - 5x + 4}{2x - 3} = 0;$$

consequently,

$$\lim_{x \rightarrow 1} \frac{2x - 3}{x^2 - 5x + 4} = \infty.$$

(8) Find $\lim_{x \rightarrow \frac{1}{2}} \frac{4x^2 - 4x + 1}{4x^2 - 1}.$

The denominator and numerator are both infinitesimals. But, by observing that

$$4x^2 - 4x + 1 = (2x - 1)^2, \quad 4x^2 - 1 = (2x - 1)(2x + 1),$$

we get

$$\lim_{x \rightarrow \frac{1}{2}} \frac{4x^2 - 4x + 1}{4x^2 - 1} = \lim_{x \rightarrow \frac{1}{2}} \frac{2x - 1}{2x + 1} = \frac{0}{2} = 0.$$

(9) When x tends arbitrarily to infinity, a linear rational function either tends to zero, or to infinity, or to a finite non-zero number, depending on whether the power of the numerator is less than, greater than, or equal to the power of the denominator.

For, given the rational fraction

$$\frac{a_0 x^m + a_1 x^{m-1} + \dots + a_m}{b_0 x^n + b_1 x^{n-1} + \dots + b_n},$$

let us divide all the terms of numerator and denominator by x^n :

$$\frac{a_0 x^{m-n} + a_1 x^{m-1-n} + \dots + a_m x^{-n}}{b_0 + b_1 x^{-1} + \dots + b_n x^{-n}};$$

as $x \rightarrow \infty$, the denominator tends to b_0 , and the numerator to zero if $m < n$, to ∞ if $m > n$, and to a_0 if $m = n$.

34. Tests for the existence of a limit. We shall give two tests for the existence of a limit.

First test. If the value of a function $f(x)$ lies between the corresponding values of two functions $F(x)$ and $\Phi(x)$, which tend to the same limit A when $x \rightarrow x_0$ (or when $x \rightarrow \infty$), $f(x)$ also has a limit, equal to A .

Proof. Let

$$F(x) \leq f(x) \leq \Phi(x) \quad \text{and} \quad \lim_{x \rightarrow x_0} F(x) = \lim_{x \rightarrow x_0} \Phi(x) = A.$$

We shall prove that

$$\lim_{x \rightarrow x_0} f(x) = A.$$

We take an arbitrary ε -neighbourhood of the number A . There exists by hypothesis a δ -neighbourhood of the point x_0 such that corresponding values of $F(x)$ and $\Phi(x)$ belong to the ε -neighbourhood of A . But now, by virtue of the above inequalities, the values of $f(x)$ corresponding to points of this δ -neighbourhood of x_0 will also lie in the ε -neighbourhood. Thus, for any ε we can find a δ such that, with $0 < |x_0 - x| < \delta$, we have $|A - f(x)| < \varepsilon$, i.e.

$$\lim_{x \rightarrow x_0} f(x) = A.$$

The proof is similar if $x \rightarrow \infty$.

If $f(x)$ is a constant, we find that this constant is equal to the common limit A of functions $F(x)$ and $\Phi(x)$.

Second test. Let x tend to x_0 (or to ∞) whilst varying in accordance with some law—e.g. only increasing or only decreasing—and let the function $f(x)$ vary monotonically in this case; then:

(1) if $f(x)$ increases but remains less than a number M , the function has a limit A , not greater than M :

$$\lim_{\substack{x \rightarrow x_0 \\ (x \rightarrow \infty)}} f(x) = A \leq M;$$

(2) if $f(x)$ decreases, but remains greater than a number M , the function has a limit A , not less than M :

$$\lim_{\substack{x \rightarrow x_0 \\ (x \rightarrow \infty)}} f(x) = A \geq M;$$

(3) if $f(x)$ is an unbounded function, it tends to positive (if increasing) or negative (if decreasing) infinity:

$$\lim_{\substack{x \rightarrow x_0 \\ (x \rightarrow \infty)}} f(x) = \pm \infty.$$

Suppose that $f(x)$ increases for the given variation of x . Obviously, two cases are possible: (1) $f(x)$ becomes greater than any previously assigned positive number (the point $y = f(x)$ moves in one direction—upwards along Oy —and goes outside any neighbourhood of $y = 0$),

so that $f(x) \rightarrow +\infty$; (2) $f(x)$ increases but remains less than a number M (the point $y = f(x)$ moves in one direction—upwards along Oy —but does not go outside the boundaries of some interval); then $f(x)$ must tend to a limit A , which does not exceed M . The strict proof of this proposition is based on the theory of real numbers and is not given here.

The situation is similar when $f(x)$ is always decreasing.

A clear picture should be formed of the fact that the “behaviour in the limit” of a function can in general be of three types: (1) either the function has a limit; (2) or it tends to infinity; (3) or finally, it neither tends to infinity nor to a limit, but oscillates constantly. The second test in fact amounts to asserting that the monotonicity of the function excludes the possibility of the third type of “behaviour in the limit”.

By virtue of this simple test, we can sometimes verify the existence of a limit (though the test does not in itself enable us to find what the limit is).

We shall consider an important example. Suppose we have the sequence

$$1, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \dots, \frac{1}{n!}, \dots,$$

where $n!$ denotes the product $1.2.3 \dots n$ of all the consecutive positive integers up to and including n (it is called “factorial n ”). We write s_n for the sum

$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}.$$

We shall prove that the sequence $s_1, s_2, \dots, s_n, \dots$ has a limit. We notice first of all that the sum s_n increases as n increases:

$$s_1 < s_2 < \dots < s_n < \dots$$

Then, since $1/k! = 1/1.2.3 \dots k < 1/1.2.2 \dots 2 = 1/2^{k-1}$, we have

$$s_n < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}$$

i.e. s_n is certainly less than $2 +$ the sum of the infinite decreasing geometrical progression $1/2 + 1/2^2 + \dots$, i.e. we have for any n : $s_n < 2 + 1 = 3$.

Hence s_n , being a bounded and increasing function, has a limit s , which is obviously greater than 2 and less than 3. The limit s is in

fact taken as the "sum" of the sequence:

$$s = \lim_{n \rightarrow \infty} s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots$$

Roughly, $s \approx 2.72$. We remarked on this special number in Chapter I (Sec. 22). It will often be encountered in what follows.

3. Continuous Functions

35. Continuity of a function. One of the chief features in the behaviour of functions is the property known as continuity. It reflects mathematically the general trait of many phenomena observed by us in nature. For instance, we speak of the continuous expansion of a rod on heating, of the continuous growth of an organism, of a continuous flow, of a continuous variation of atmospheric temperature etc.

We can say descriptively that a function is continuous if it varies "gradually", i.e. if "small" changes in the argument produce "small" changes in the function. We put the term "small" in quotation marks because it does not express a clear arithmetical concept.

Let Δx denote the increment of the argument x_0 , and Δy the corresponding increment of the function $y = f(x)$, i.e.

$$\Delta f = f(x_0 + \Delta x) - f(x_0)$$

(see Sec. 18).

On the basis of the concept of limit, we now give a strict definition of continuity of a function.

Definition. A function $y = f(x)$ is said to be *continuous at a point* x_0 (or at $x = x_0$) if it is defined in any neighbourhood of x_0 and if

$$\lim_{\Delta x \rightarrow 0} \Delta y = 0$$

or, what amounts to the same thing,

$$\lim_{\Delta x \rightarrow 0} [f(x_0 + \Delta x) - f(x_0)] = 0, \quad (\dagger)$$

i.e. if an infinitesimal increment of the function corresponds to an infinitesimal increment of the argument.

Relationship $(\dagger)$ implies that, given any positive ε , there exists a positive δ such that, for all Δx satisfying $|\Delta x| < \delta$, we have $|\Delta y| < \varepsilon$, i.e. to all values $x = x_0 + \Delta x$ sufficiently "close" to x_0 , there correspond values $f(x) = f(x_0 + \Delta x)$ as "close" as desired to $f(x_0)$.

Let us give this property a visual interpretation. We imagine the graph of the function to be made out of wire (Fig. 44). We take a hollow circular cylindrical sleeve of diameter 2ε , where ε is a previously assigned number, and place it so that the mid-point of the

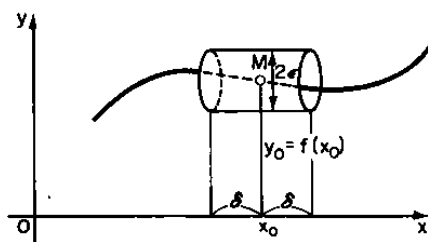


FIG. 44

axis of the sleeve is at the point $M(x_0, f(x_0))$, whilst the axis itself is parallel to Ox . We can now choose a sleeve length 2δ (which may be very small) so that the part of the wire (the graph of the function) corresponding to the interval $[x_0 - \delta, x_0 + \delta]$ will lie wholly inside the sleeve without deformation. It is clear that, the smaller the given sleeve diameter, the shorter in general the sleeve ("the smaller ε , the smaller δ ").

Limit relationship ($\dagger$) can be written as:

$$\lim_{\Delta x \rightarrow 0} f(x_0 + \Delta x) = f(x_0).$$

If we write x for $x_0 + \Delta x$, x will tend to x_0 as $\Delta x \rightarrow 0$, and the last equation can be rewritten as

$$\lim_{x \rightarrow x_0} f(x) = f(x_0).$$

This leads to a rather different statement of the continuity of a function at a point.

Definition. A function is said to be *continuous at a point* x_0 if it is defined in any neighbourhood of this point and if the limit of the function, as the independent variable x tends to x_0 , exists and is equal to the value of the function at $x = x_0$.

Thus three requirements have to be satisfied for the continuity of a function $y = f(x)$ at $x = x_0$:

- (1) $f(x)$ must be defined in any neighbourhood of x_0 ;

- (2) $f(x)$ must have a limit when x tends arbitrarily* to x_0 ;
 (3) this limit must coincide with the value of the function at x_0 .

Definition. A function which is continuous at every point of an interval is said to be *continuous in this interval*.

The continuity of a function $y = f(x)$ means geometrically that the ordinates of its graph, corresponding to two points of Ox , differ from each other by as little as desired, if the distance between the points of Ox is sufficiently small. Thus the graph of a continuous function is a continuous curve without breaks; if it can actually be traced, this can be done, on moving in one direction, say from left to right, without removing the pencil from the graph. (We shall encounter further on examples of continuous functions where it is impossible to conceive how the graph could in fact be completely traced (Fig. 45 and Fig. 62).)

It may be noticed that the limit equation $\lim_{x \rightarrow x_0} f(x) = f(x_0)$, expressing by definition the continuity of the function at $x = x_0$, can be rewritten as:

$$\lim_{x \rightarrow x_0} f(x) = f(\lim_{x \rightarrow x_0} x). \quad (\dagger\dagger)$$

Thus, if the symbol of the limit and the symbol of the function can be interchanged, the function is continuous at the limiting value of the argument.

If a function is known to be continuous at the limit point, equation $(\dagger\dagger)$ expresses the following important rule for passage to the limit:

To find the limit of a continuous function, we only have to replace the argument in the expression for the function by its limiting value.

We showed in Sec. 33 (example 5) that $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ if $f(x)$ is a linear rational function. Thus a linear rational function is continuous at every point of its domain of definition. In particular, *any polynomial is continuous throughout Ox .*

36. Points of discontinuity of a function.

Definition. The point $x = x_0$ is called a *point of discontinuity* of a function $y = f(x)$ if $f(x)$ is defined in some neighbourhood of the

* If x_0 coincides with the end of the interval in which $f(x)$ is defined, the statement of the first two requirements must be slightly modified: we must only consider values of x lying to the right of x_0 (if x_0 is the left-hand end), or to the left of x_0 (if x_0 is the right-hand end), and similarly, x must tend to x_0 from the right (or from the left).

point x_0 except for the point itself or if $f(x)$ is defined at x_0 itself, but requirement (2) or (3) of Sec. 35 is not fulfilled.

In other words:

Either $f(x)$ has no limit when x tends arbitrarily to x_0 , or this limit exists but does not coincide with the value of the function at x_0 .*

A function $y = f(x)$ which has a discontinuity at the point x_0 is said to be discontinuous at this point; we can also say that $y = f(x)$ has a break or jump at $x = x_0$.

We call $x = x_0$ a point of infinite discontinuity of the function if, in the neighbourhood of $x = x_0$, the function is unbounded (and, in particular, is infinitely large); we say that the function has an infinite "jump" as the argument x passes through x_0 . Otherwise the discontinuity is called finite.

Let the left and right-hand limits of the function $f(x)$ exist at the point x_0 , i.e. $f(x_0 - 0)$ and $f(x_0 + 0)$ exist (see Sec. 29).

If

$$f(x_0 - 0) \neq f(x_0 + 0) \quad \text{or} \quad f(x_0 - 0) = f(x_0 + 0) \neq f(x_0),$$

x_0 is obviously a point of discontinuity; in this case it is termed a point of discontinuity of the first kind.

Definition. A point of discontinuity of the first kind of a function $f(x)$ is a point of discontinuity at which $f(x)$ has left and right-hand limits.

Every point of discontinuity not of the first kind is called a point of discontinuity of the second kind.

A function has a finite jump at a point of discontinuity of the first kind.

We shall give some examples of discontinuous functions.

(1) The function $y = 1/x$ is discontinuous at the point $x = 0$, since the function is not defined for this value of x (the discontinuity is infinite). If x tends to zero from the left, the function tends to $-\infty$, whilst if from the right, to $+\infty$.

The function $y = \tan x$ (Fig. 30) is discontinuous at the points $x = (2k + 1)\pi/2$, where k is any integer, since the function is not defined with these values of x (here again the discontinuities are infinite).

* The points of discontinuity of a function also include points (which may not in fact satisfy our definition) which are "limits" for the set of points of discontinuity of the function. An example is the point $x = 0$ for the function $y = 1/(\sin 1/x)$. We shall leave these more complicated cases on one side.

If x tends to $(2k + 1)\pi/2$ from the left, the function tends to $+\infty$, and if from the right, to $-\infty$.

The point $x = 0$ is a point of discontinuity (finite) of the function $y = \sin 1/x$, but unlike the first two examples, $y = \sin 1/x$ has no limit, however x tends to zero—according to an arbitrary law, or only from the right or only from the left; y constantly oscillates between -1 and $+1$ (Fig. 45).

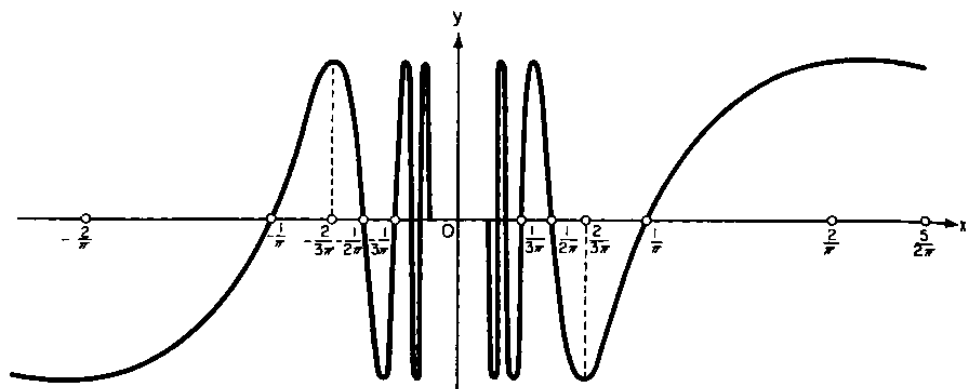


FIG. 45

All the above are examples of points of discontinuity of the second kind.

(2) The function $y = f(x)$, defined for any $x \neq 0$ by the formula

$$y = f(x) = \frac{1}{1 + 2^{\frac{1}{x}}},$$

and equal to zero for $x = 0$: $f(0) = 0$, has a (finite) discontinuity at the point $x = 0$, because the second of the conditions for continuity is not fulfilled: the function has no limit when the argument x tends arbitrarily to zero. For, let x tend to zero from the right, i.e. remaining positive; then $1/x \rightarrow +\infty$, $2^{1/x} \rightarrow \infty$ and

$$\lim_{x \rightarrow 0} \frac{2}{1 + 2^{\frac{1}{x}}} = 0, \quad \text{i.e.} \quad f(0+0) = 0;$$

whereas if x tends to zero from the left, i.e. remaining negative, $1/x \rightarrow -\infty$, $2^{1/x} \rightarrow 0$ and

$$\lim_{x \rightarrow 0} \frac{1}{1 + 2^{\frac{1}{x}}} = 1, \quad \text{i.e.} \quad f(0-0) = 1.$$

The function thus has different limits when x tends to zero from the right only or from the left only (i.e. different left- and right-hand limits). This indicates the absence of a limit when x tends arbitrarily to zero, i.e. $x = 0$ is a point of discontinuity of the first kind.

The graph of this function has the form shown in Fig. 46.

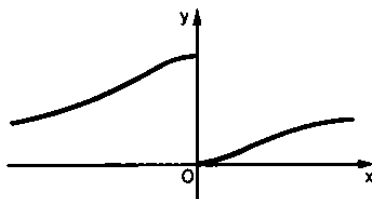


FIG. 46

Whatever value we choose at $x = 0$ for the function, instead of $y = 0$, it always remains impossible to arrange for continuity of the function at $x = 0$, since it has no limit as $x \rightarrow 0$. The discontinuity is retained at $x = 0$ for any value of $f(0)$.

(3) We define a function $y = f(x)$ as follows: $f(x)$ is equal to x for all x except $x = 1$; its value is equal to $\frac{1}{2}$ for $x = 1$. We write this as:

$$f(x) = \begin{cases} x, & x \neq 1, \\ \frac{1}{2}, & x = 1. \end{cases}$$

Our function $f(x)$ is discontinuous at $x = 1$. For it has a limit, equal to 1, as x tends to 1, and this limit is not equal to $\frac{1}{2}$, as would be required for continuity. The graph of the function (Fig. 47) consists of all the straight line $y = x$ except for the point $(1, 1)$, instead of which the graph has the point $(1, \frac{1}{2})$. For this function, $f(1 - 0) = f(1 + 0) \neq f(1)$, i.e. $x = 1$ is a point of discontinuity of the first kind.

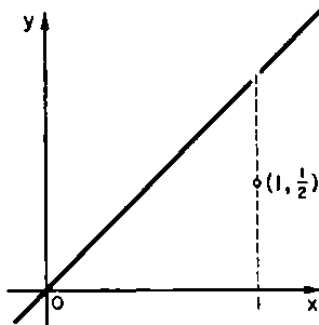


FIG. 47

If $\lim f(x)$ exists with x tending to x_0 in an arbitrary manner, we say in cases where the limit is not equal to $f(x_0)$, i.e. where $f(x_0 - 0) = f(x_0 + 0) \neq f(x_0)$, or where $f(x)$ is not defined at x_0 , that the point of discontinuity x_0 is a "point of removable discontinuity" ($f(x)$ has a "removable discontinuity" at x_0), whilst the search for the limit is sometimes called removal of the discontinuity. If we find $\lim_{x \rightarrow x_0} f(x)$ and take this number as the value of $f(x_0)$, we so to speak "remove" the discontinuity at x_0 , making the function continuous at this point. (For instance, the function in example (3) has a removable discontinuity at $x = 1$).

Suppose a function has a removable discontinuity at the point x_0 . The limit of the function as the independent variable tends to this point is sometimes called by tradition the "true value" of the function at the point, whilst the process of finding the limit is called "removal of an indeterminacy".

For instance, we say that $1/6$ is the "true value" of the function $y = (x - 3)/(x^2 - 9)$ at $x = 3$. If we take a new function, coinciding with $y = (x - 3)/(x^2 - 9)$ for all x different from 3, and equal to $1/6$ at $x = 3$, it proves to be not only defined, but also continuous everywhere, including at $x = 3$.

Another example: the function $y = x \sin 1/x$ is not defined for $x = 0$, but tends to zero as $x \rightarrow 0$, since $|x \sin 1/x| = |x| \left| \sin \frac{1}{x} \right| \leq |x| \cdot 1 = |x|$; hence another function, specified as: $y = x \sin \frac{1}{x}$ if $x \neq 0$, and $y = 0$ if $x = 0$, is defined and continuous for all x .

We shall not follow this terminology, however, since we can assign with the same justice to a function, not defined at $x = x_0$, a value different from its limit as $x \rightarrow x_0$, and generally speaking, there is no sort of basis for assuming that one of these values is a "true" value and another is not "true".

The concept of function covers such a wide field that we can conceive of discontinuous functions of a very complicated structure, for instance, those which are defined and discontinuous at every point of the real axis*. In natural science and technology, however, we are concerned in the majority of cases with continuous functions. Similarly, the functions which will be most often encountered in this book, viz. the elementary functions (see Sec. 12) are continuous wherever they are defined. We shall discuss this point in Sec. 39.

* An example is the function equal to zero at all irrational points and equal to unity at all rational points. The domain of definition of this function is the entire real axis, whilst every point of the axis is a point of discontinuity of the function. It is curious that several different analytic expressions (including a passage to the limit) have been constructed that accurately express such an extremely "complicated" and "freakish" function.

37. General properties of continuous functions. The assumption that a function is continuous at every point of a closed interval (i.e. is continuous in a closed interval) implies in itself that the function must have several important properties of a general kind*. We shall mention some of these properties.

THEOREM I. A function which is continuous in a *closed* interval achieves a maximum and minimum value at least once in this interval.

Let $y = f(x)$ be a continuous function in the interval $[a, b]$ (Fig. 48). The theorem asserts the existence in the interval $[a, b]$ of at least one point ξ_1 , $a \leq \xi_1 \leq b$, and at least one point ξ_2 , $a \leq \xi_2 \leq b$, such

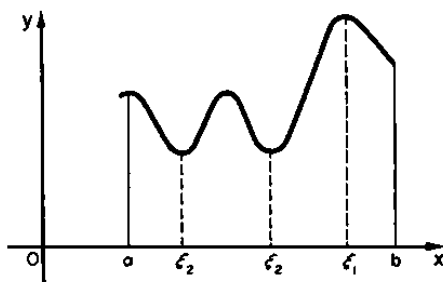


FIG. 48

that the values of the function at these points are respectively the greatest and least of its values throughout $[a, b]$:

$$\left. \begin{array}{l} f(x) \leq f(\xi_1) \\ f(x) \geq f(\xi_2) \end{array} \right\} \text{ for all } x, a \leq x \leq b.$$

This assertion does not in general hold, unless we assume that the interval is closed. For instance, the function $y = x$ is continuous in any open interval (a, b) , but attains neither a maximum nor minimum in it: it attains these at the ends, which are excluded from the interval. Similarly, the theorem ceases to be true for discontinuous functions. The reader should think of an example of a function (at any rate given graphically), which is defined and discontinuous in a closed interval, and does not satisfy theorem I.

THEOREM II. A function which is continuous in a *closed* interval and takes values of differing sign at the ends of the interval, vanishes at least once inside the interval.

* We recommend the reader, desirous of an acquaintance with the proofs of the following three theorems, to a more complete course of analysis.

Let $f(x)$ be continuous in $[a, b]$ and let $f(a) < 0$, $f(b) > 0$ (or $f(a) > 0$, $f(b) < 0$). The theorem says that there exists at least one point ξ inside $[a, b]$, $a < \xi < b$, such that $f(\xi) = 0$. This implies geometrically the obvious fact that a continuous curve—the graph of the continuous function—joining a point A , lying below (above) Ox , to a point B , lying above (below) Ox , must cut Ox in at least one point (Fig. 49). In general, discontinuous functions do not possess this property.

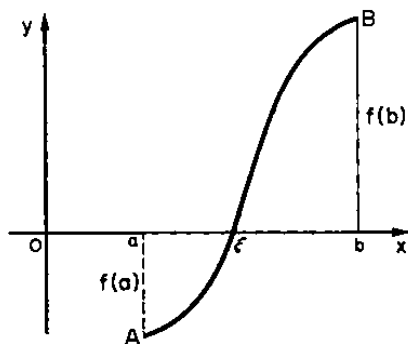


FIG. 49

Theorem II can be stated in a more general form:

A function, continuous in a closed interval, takes at least once inside the interval any value lying between its values at the ends of the interval.

Let $f(a) = m$, $f(b) = M$, and let say $m < M$ (the assumption that $m > M$ leads to only trivial changes in the argument). Let μ be any number lying between m and M : $m < \mu < M$. The proposition asserts the existence in the interval $[a, b]$ of at least one point ξ , such that $f(\xi) = \mu$. This proposition soon reduces to theorem II. In fact, we take the auxiliary function

$$\varphi(x) = f(x) - \mu;$$

this takes values of differing sign at the ends of the interval:

$\varphi(a) = f(a) - \mu = m - \mu < 0$ and $\varphi(b) = f(b) - \mu = M - \mu > 0$. Since $\varphi(x)$ is a continuous function in $[a, b]^*$, it vanishes by theorem II

* This is directly obvious, but a simple proof may be given. We have

$$\begin{aligned}\Delta \varphi(x) &= \varphi(x + \Delta x) - \varphi(x) = [f(x + \Delta x) - \mu] - [f(x) - \mu] \\ &= f(x + \Delta x) - f(x) = \Delta f(x),\end{aligned}$$

and since $\Delta f(x) \rightarrow 0$ as $\Delta x \rightarrow 0$ by hypothesis, $\Delta \varphi(x) \rightarrow 0$ also, which shows, in fact, that $\varphi(x)$ is a continuous function.

at some ξ lying between a and b , i.e. $\varphi(\xi) = 0$. Hence $f(\xi) = \mu = 0$, i.e. $f(\xi) = \mu$.

This proposition means geometrically that any straight line $y = \mu$, parallel to Ox and passing between the initial and final points of a continuous curve, cuts the curve at least once. The abscissa of the point of intersection will be the value $x = \xi$ for which $f(\xi) = \mu$ (Fig. 50). This proposition shows that:

When passing from one value to another, a continuous function must take at least once all intermediate values; in particular:

A function, continuous in an interval, takes any value lying between its maximum and minimum values at least once in the interval.

THEOREM III. If a function $y = f(x)$ is continuous in a closed interval $[a, b]$, given any positive ε , a positive δ can be found such that

$$|f(x + \Delta x) - f(x)| < \varepsilon,$$

provided

$$|\Delta x| < \delta$$

independently of the value of x , $a \leq x \leq b$.

The function $f(x)$ in fact has this property at every point of $[a, b]$ separately, since the property is inherent, by definition, in a function continuous at a given point (Sec. 35). With a given value of x , for any $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that $0 < |\Delta x| < \delta$ implies $|\Delta y| = |f(x + \Delta x) - f(x)| < \varepsilon$. But we cannot conclude from this that, given any ε , a δ can be chosen which is suitable for any point of the interval. The argument here might run as follows. If we take two, three, and in general a finite number n of points $x_1, x_2, \dots, x_n$, of the interval, we evidently obtain n values for δ , corresponding to the same ε ($\delta_1, \delta_2, \dots, \delta_n$), among which the least—call it δ_m —will be suitable for any of the chosen points; thus the value of $f(x)$ at an arbitrary point of a δ_m -neighbourhood of any of our points x_i will differ (in absolute value) from the value of $f(x_i)$ by less than ε . But if we take an infinite sequence of points $x_1, x_2, \dots, x_n, \dots$ of the interval, we cannot deny in advance the possibility of a case when the required δ diminishes on passing from one point x_n to the next x_{n+1} , and tends to zero as $n \rightarrow \infty$; in which case, there will be no common δ for all the points considered. Still less can we be assured in advance that such a common δ exists for any $\varepsilon > 0$ for all points of interval $[a, b]$.

Yet theorem III in fact asserts that there can be no such cases, and that we can always find a δ , given the same ε , which is suitable

for all points of a closed interval in which the function is continuous. This means that all we need do, in order to achieve a given degree of closeness between two values of the function, is to assure the same degree of closeness between corresponding values of the argument independently of where these values are actually situated in the closed interval. This fact testifies to the "equal validity", "sameness" or "uniformity" of all parts of the interval as regards the choice

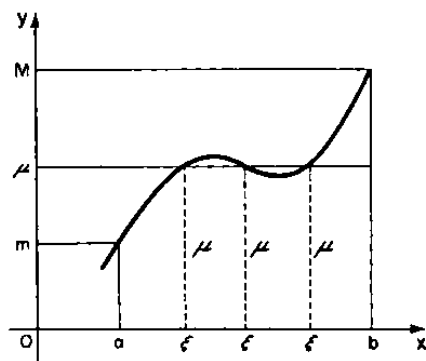


FIG. 50

of δ for a given ε ; the property of functions, stated in theorem III, is therefore called uniform continuity. Theorem III can thus also be stated as:

A function which is continuous in a *closed* interval is uniformly continuous in it.

A clear picture of the property of uniform continuity can be obtained with the aid of the interpretation of Sec. 35 of the property of continuity (see Fig. 44). According to theorem III, given a diameter of 2ε of the cylindrical sleeve, we can choose its length 2δ (which may be extremely small) such that the sleeve can move from one end to the other along the wire representing the graph of the function without touching the wire; here, the centre of the sleeve axis always remains on the wire, and its axis always remains parallel to the axis of the independent variable.

The property of uniform continuity of functions plays a big role in various problems of mathematical analysis.

It may easily be shown that theorem III is in general untrue for a non-closed interval. For instance, the function $y = 1/x$ is continuous in the semi-closed interval $(0, 1]$, but this continuity is not uniform. For, given any $\varepsilon > 0$, whatever $\delta > 0$ we choose we can find a point

$x_0 > 0$ such that the continuity condition is not fulfilled. We must have, for such an x_0 :

$$\frac{1}{x_0} - \frac{1}{x_0 + \delta} > \varepsilon,$$

or

$$\frac{\delta}{x_0(x_0 + \delta)} > \varepsilon,$$

and this inequality (allowing for the fact that we can take $x_0 < 1$) will certainly hold if

$$\frac{\delta}{x_0 + \delta} > \varepsilon, \quad \text{i.e.} \quad \frac{x_0 + \delta}{\delta} < \frac{1}{\varepsilon},$$

whence

$$x_0 < \delta \left(\frac{1}{\varepsilon} - 1 \right).$$

Given ε and δ , we can therefore always find the required $x = x_0$.

38. Operations on continuous functions. Continuity of the elementary functions.

I. Suppose we have a generally known stock of functions, from which other functions are formed by means of the arithmetic operations and operations of taking a function of a function, a finite number of all these operations being involved. (We recall that the elementary functions are in fact formed in this manner from the basic elementary functions.)

We shall prove that:

If the original functions are continuous, the functions formed from them in the manner described will, in general, also be continuous.

This proposition is a consequence of the next four theorems, which assert the continuity of the functions obtained as a result of applying each of our operations to a continuous function.

THEOREM I. The (algebraic) sum of two, three and in general any definite finite number of functions, continuous at a point, is also a continuous function at that point.

Proof. Let $u, v, \dots, t$, be a definite number of functions continuous at the point $x = x_0$. We want to show that their sum $w = u + v + \dots + t$ is also a continuous function at $x = x_0$. Since the functions are continuous, we have

$$\lim_{x \rightarrow x_0} u = u_0, \quad \lim_{x \rightarrow x_0} v = v_0, \quad \dots, \quad \lim_{x \rightarrow x_0} t = t_0,$$

where $u_0, v_0, \dots, t_0$ are the values of functions $u, v, \dots, t$ at $x = x_0$. We have by the theorem on the limit of a sum (Sec. 32):

$$\begin{aligned}\lim_{x \rightarrow x_0} w &= \lim_{x \rightarrow x_0} (u + v + \dots + t) = \lim_{x \rightarrow x_0} u + \lim_{x \rightarrow x_0} v + \dots + \lim_{x \rightarrow x_0} t \\ &= u_0 + v_0 + \dots + t_0 = w_0,\end{aligned}$$

where w_0 is the value of function w at $x = x_0$, and this in fact proves the continuity of w at $x = x_0$.

THEOREM II. The product of two, three and in general any finite number of functions, continuous at a point, is a continuous function at that point.

Proof. Retaining the notation of theorem I and taking $w = u \cdot v \dots t$, we have by the theorem on the limit of a product (Sec. 32):

$$\begin{aligned}\lim_{x \rightarrow x_0} w &= \lim_{x \rightarrow x_0} (u \cdot v \dots t) = \lim_{x \rightarrow x_0} u \cdot \lim_{x \rightarrow x_0} v \dots \lim_{x \rightarrow x_0} t \\ &= u_0 \cdot v_0 \dots t_0 = w_0.\end{aligned}$$

This is what we wanted to show.

THEOREM III. The quotient of two functions, continuous at a point, is a continuous function at that point, provided the denominator does not vanish at it.

Proof. As above, taking $w = u/v$, we have by the theorem on the limit of a quotient (Sec. 32):

$$\lim_{x \rightarrow x_0} w = \lim_{x \rightarrow x_0} \frac{u}{v} = \frac{\lim_{x \rightarrow x_0} u}{\lim_{x \rightarrow x_0} v} = \frac{u_0}{v_0} = w_0.$$

This is what we had to prove.

The theorem on the limit of a quotient is only valid if $\lim v = v_0 \neq 0$.

THEOREM IV. A function of a function, formed from a finite number of continuous functions, is continuous.

Proof. We only need to prove this theorem for a function chain with two links, then it can be extended successively to a chain with any finite number of links.

Let $y = f(z)$, whilst $z = \varphi(x)$, so that

$$y = f[\varphi(x)] = F(x),$$

$\varphi(x)$ being continuous at $x = x_0$, whilst $f(z)$ is continuous at $z = z_0$, where $z_0 = \varphi(x_0)$. The theorem says that y , regarded as a function of x , i.e. $F(x)$, is continuous at $x = x_0$.

We have in fact, by virtue of the fact that $z \rightarrow z_0$ as $x \rightarrow x_0$ and the continuity of the function f :

$$\lim_{x \rightarrow x_0} y = \lim_{z \rightarrow z_0} f(z) = f(z_0),$$

but

$$z_0 = \varphi(x_0),$$

i.e.

$$\lim_{x \rightarrow x_0} y = \lim_{x \rightarrow x_0} F(x) = f[\varphi(x_0)] = F(x_0).$$

This is what we had to show.

THEOREM V. The inverse of a monotonic and continuous function is a continuous function.

We omit the proof of this theorem.

II. In view of what was said in I, the continuity of the elementary functions follows from the continuity of the functions composing the initial stock, i.e. from the continuity of the basic elementary functions. Properly speaking, in all our previous discussion, and in particular in the description of the properties and graphs of the basic elementary functions in Chapter I, we have already implicitly assumed that these functions are continuous (the same assumption is also made in elementary mathematics).

The aim and scope of the present book compel us to omit a discussion of the continuity of the basic elementary functions*.

We shall acquaint the reader with a proof only of the continuity of the trigonometric and inverse trigonometric functions; this can easily be done directly from the geometrical definitions of these functions.

THEOREM. *Each of the trigonometric functions $y = \sin x$, $y = \cos x$, $y = \tan x$ etc. is continuous at any point x at which it is defined.*

Proof. We start with the function $y = \sin x$. We show that

$$\lim_{\Delta x \rightarrow 0} [\sin(x + \Delta x) - \sin x] = 0$$

or any x . We have

$$\lim_{\Delta x \rightarrow 0} [\sin(x + \Delta x) - \sin x] = 2 \lim_{\Delta x \rightarrow 0} \left[\sin \frac{\Delta x}{2} \cdot \cos \left(x + \frac{\Delta x}{2} \right) \right].$$

If $\Delta x \rightarrow 0$, then $\frac{1}{2}\Delta x \rightarrow 0$, and we shall show that now, $\sin \frac{1}{2}\Delta x$ also tends to zero. The required limit equation will follow from this, since the second factor— $\cos(x + \frac{1}{2}\Delta x)$ —is a bounded function.

* As before, we recommend the reader interested in a detailed account to refer to a more complete course of analysis.

We shall prove the general validity of the relationship

$$\lim_{\alpha \rightarrow 0} \sin \alpha = 0,$$

implying, by virtue of the equation $\sin 0 = 0$, that the function $y = \sin \alpha$ is continuous at $\alpha = 0$. Since $\alpha \rightarrow 0$, we can assume that $|\alpha| < \frac{1}{2}\pi$. We take a circle of unit radius and draw two radii on either side of a given, say horizontal, radius, each at an angle α from it. The chord joining the ends of the two radii is equal by definition to $2 \sin \alpha$, whilst the arc of the circle subtending this chord is equal to 2α . It is clear that the chord is less than the arc, so that

$$2|\sin \alpha| < 2|\alpha| \quad \text{or} \quad |\sin \alpha| < |\alpha|;$$

hence, if $\alpha \rightarrow 0$, also $\sin \alpha \rightarrow 0$. This is what we had to show.

The function $y = \cos x$ is continuous for any x , because the functions $y = \sin z$ and $z = x + \frac{1}{2}\pi$, forming the function of a function $y = \sin(x + \frac{1}{2}\pi) = \cos x$ are continuous.

Similarly, the function $y = \tan x = \sin x / \cos x$, by virtue of the continuity of the numerator and denominator, is continuous for any x , except those for which $\cos x = 0$, i.e. except for $x = (2n + 1)\pi/2$, where n is any integer or zero.

The proofs of the continuity of the other trigonometric functions are similar.

THEOREM. *The inverse trigonometric functions $y = \arcsin x$, $y = \arccos x$, $y = \arctan x$ and so on are continuous at any point x where they are defined.*

Proof. Each of the inverse trigonometric functions (strictly speaking: their principal branches) is the inverse of a continuous and monotonic function, and is therefore (theorem V) itself a continuous and monotonic function. (For instance, $y = \arcsin x$ is the inverse of the function $y = \sin x$, which is continuous and increasing in the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$.)

An elementary function can only have discontinuities for values of the independent variable at which certain of the elements forming the function become indeterminate, or at which the denominators of fractions entering into the expression for the function vanish.

Therefore, *the interval of continuity of an elementary function precisely coincides with its interval of definition.*

A simple, convenient practical rule for passage to the limit follows from this:

To find the limit of an elementary function, when the argument tends to a value belonging to the domain of definition of the function, we have to replace the argument in the expression for the function by its limiting value.

This rule is extremely important, since elementary functions are used most frequently in the applications of analysis.

Cases when the argument tends to infinity or to some value representing a point of indeterminacy of the function always have to be specially considered.

4. Comparison of Infinitesimals. Some Important Limits

39. Comparison of infinitesimals. Equivalent infinitesimals.

I. THE ORDER OF AN INFINITESIMAL. In order to compare two infinitesimals, we take the limit of their ratio.

Let α and β be infinitesimals and let $\lim \alpha/\beta = 0$. This means that α tends to zero, not only when its "proper" value is measured in terms of a constant unit, but also when its "relative" value is measured in terms of an infinitesimal variable "unit" β . We say in this case that α is an infinitesimal of higher order than β . We can also say that α tends to zero more "rapidly" than β .

To take an example, let $\alpha_n = 1/n^2$, and $\beta_n = 1/n$ ($n \rightarrow \infty$). We find on comparing these infinitesimals, that α_n is of higher order than β_n , since $\alpha_n = \beta_n^2$, and

$$\lim_{n \rightarrow \infty} \frac{\alpha_n}{\beta_n} = \lim_{n \rightarrow \infty} \frac{\beta_n^2}{\beta_n} = \lim_{n \rightarrow \infty} \beta_n = 0.$$

If α is an infinitesimal of higher order than β , we can say alternatively that β is an infinitesimal of lower order than α . Now

$$\lim \frac{\beta}{\alpha} = \infty.$$

Definition. α is an infinitesimal of higher or lower order than β , or of the same order, according to whether $\lim \alpha/\beta$ is zero, infinity, or a finite non-zero number.

If we regard zero as an infinitesimal, no other infinitesimal can be compared with it, inasmuch as it is not permissible to divide by zero*.

The comparison of two infinitesimals is generalized as follows: an infinitesimal α is of the k -th order of smallness with respect to an infinitesimal β if α/β^k tends to a non-zero limit.

If $k > 1$, α is an infinitesimal of higher order than β , whilst if $k < 1$, it is of lower order. For, since $\alpha/\beta^k \rightarrow B \neq 0$, i.e. $(\alpha/\beta)/\beta^{k-1} \rightarrow B$, in the first case α/β must tend to zero (otherwise $(\alpha/\beta) \cdot (1/\beta^{k-1})$ could not tend to a limit), and in the second case α/β must tend to ∞ (otherwise $(\alpha/\beta) \cdot \beta^{1-k}$ could not tend to a non-zero number).

When we are concerned with several infinitesimals simultaneously, one is usually chosen as the "unit of measurement", and all the rest are compared with it.

* Also, two infinitesimals α and β cannot be compared if their ratio has no limit.

II. THE EQUIVALENCE OF INFINITESIMALS.

Definition. If α and β are infinitesimals of the same order, whilst

$$\lim \frac{\alpha}{\beta} = 1,$$

they are said to be equivalent. In this case, $\lim \beta/\alpha = 1$ also.

The following simple but important proposition holds.

THEOREM. If α and β are equivalent infinitesimals, the difference between α and β is an infinitesimal of higher order than α or β .

Proof. Writing $\alpha - \beta = \gamma$, we have:

$$\lim \frac{\gamma}{\beta} = \lim \left(\frac{\alpha}{\beta} - 1 \right) = 0,$$

since α/β tends to unity. Similarly, $\lim \gamma/\alpha = 0$.

CONVERSE. If two infinitesimals α and β differ by an infinitesimal of higher order, they are equivalent.

Proof. Let $\gamma = \alpha - \beta$ and $\gamma/\alpha \rightarrow 0$ (or $\gamma/\beta \rightarrow 0$). We have to show that $\beta/\alpha \rightarrow 1$. But $\gamma/\alpha = (\alpha - \beta)/\alpha = 1 - (\beta/\alpha)$; thus $\gamma/\alpha \rightarrow 0$ implies $\beta/\alpha \rightarrow 1$. Similarly, $\gamma/\beta \rightarrow 0$ implies $\alpha/\beta \rightarrow 1$.

The equivalence of infinitesimals α and β is sometimes written as: $\alpha \sim \beta$.

Example. The infinitesimal $\alpha_n = (n+1)/n^2$ is equivalent, when $n \rightarrow \infty$, to the infinitesimal $\beta_n = 1/n$, since they differ by a higher order infinitesimal: $\alpha_n - \beta_n = 1/n^2$. Their ratio tends to unity:

$$\frac{\alpha_n}{\beta_n} = \frac{n+1}{n} \rightarrow 1.$$

In many problems infinitesimals can simply be replaced by their equivalents* without any error. For instance, the following proposition holds:

The limit of the ratio of two infinitesimals is unchanged if they are replaced by equivalent infinitesimals.

For, let $\alpha \sim \alpha'$, $\beta \sim \beta'$. Then

$$\begin{aligned} \lim \frac{\alpha}{\beta} &= \lim \left(\frac{\alpha}{\alpha'} \cdot \frac{\alpha'}{\beta'} \cdot \frac{\beta'}{\beta} \right) = \lim \frac{\alpha}{\alpha'} \cdot \lim \frac{\alpha'}{\beta'} \cdot \lim \frac{\beta'}{\beta} \\ &= 1 \cdot \lim \frac{\alpha'}{\beta'} \cdot 1 = \lim \frac{\alpha'}{\beta'}. \end{aligned}$$

* This explains the term "equivalent" infinitesimals.

It follows from this proposition that, when seeking the limit of a ratio α/β of infinitesimals α and β , we can neglect infinitesimals of higher orders in both numerator and denominator (because, by what has been proved, this amounts to replacing infinitesimals α and β by their equivalents).

The question of approximating one function by another may be mentioned in connection with the equivalence of infinitesimals. Let two functions u and v tend to the same limit A as $x \rightarrow x_0$ (or as $x \rightarrow \infty$), i.e. $\lim u = \lim v = A$. Then $\lim(u - v) = 0$, i.e. $u - v = \alpha$ is an infinitesimal. Hence, if we replace the values of function u by the values of v in the neighbourhood of the point x_0 (or for sufficiently large (in absolute value) x) the *absolute error* α will be an infinitesimal. We know, however, that the "quality" of an approximation is judged by its *relative*, not by its absolute error. We can say that:

The values of magnitude v express the values of magnitude u "with unlimited accuracy" as $x \rightarrow x_0$ (or as $x \rightarrow \infty$) if the relative error of the approximate equation $u \approx v$ tends to zero, i.e. $\frac{\alpha}{u} \rightarrow 0$.

If $A \neq 0$, we have in fact:

$$\frac{\alpha}{u} = \frac{u - v}{u} = \left(1 - \frac{v}{u}\right) \rightarrow 1 - \frac{A}{A} = 0.$$

But when $A = 0$, i.e. when both functions u and v are infinitesimals, the relative error may not be an infinitesimal. It will be an infinitesimal only in the case when u and v are equivalent infinitesimals. For, the fact that

$$\frac{\alpha}{u} = \frac{u - v}{u} \rightarrow 0$$

implies that u and v differ by a higher order infinitesimal, i.e. they are equivalent.

Thus the condition for the values of one infinitesimal to be approximate values of the other with unlimited accuracy is that the infinitesimals be equivalent.

This may also be stated as follows:

Each of two equivalent infinitesimals is the principal part of the other.

Definition. The principal part of an infinitesimal is defined as the infinitesimal which differs from it by an infinitesimal of higher order.

For example, as $n \rightarrow \infty$ the infinitesimal $\beta_n = 1/n$ is the principal part of the infinitesimal $\alpha_n = (n + 1)/n^2$.

40. Examples of ratios of infinitesimals. Let us find, as examples, the limits of three ratios of infinitesimals. The last of these is of great importance in analysis.

I. $\lim_{x \rightarrow 0} \frac{\sqrt{x+4}-2}{\sqrt{9+x}-3}$. The function under the limit sign is not defined for $x = 0$, since the denominator vanishes. We carry out some simple algebraic transformations:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{4+x}-2}{\sqrt{9+x}-3} &= \lim_{x \rightarrow 0} \frac{(\sqrt{4+x}-2)(\sqrt{4+x}+2)(\sqrt{9+x}+3)}{(\sqrt{9+x}-3)(\sqrt{4+x}+2)(\sqrt{9+x}+3)} \\ &= \lim_{x \rightarrow 0} \frac{\sqrt{9+x}+3}{\sqrt{4+x}+2}.\end{aligned}$$

We now have to find the limit of a function continuous at $x = 0$; by the familiar rule (Sec. 38), it is equal to $\frac{\sqrt{9+0}+3}{\sqrt{4+0}+2} = 3/2$.

II. $\lim_{x \rightarrow 0} \frac{\sqrt{1+x}-1}{\frac{1}{2}x}$. We get by a similar procedure:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{1+x}-1}{\frac{1}{2}x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{\frac{1}{2}x(\sqrt{1+x}+1)} \\ &= \lim_{x \rightarrow 0} \frac{1}{\frac{1}{2}(\sqrt{1+x}+1)} = \frac{1}{\frac{1}{2} \cdot 2} = 1.\end{aligned}$$

We see that the numerator and denominator are equivalent infinitesimals as $x \rightarrow 0$: $\sqrt{1+x}-1 \sim \frac{1}{2}x$. We can therefore take approximately, for sufficiently small x :

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x$$

with as small a relative error as desired.

Let us work out say $\sqrt[3]{627}$. We have:

$$\sqrt[3]{627} = \sqrt[3]{625+2} = \sqrt[3]{625} \cdot \sqrt[3]{1+\frac{2}{625}} \approx 25 \left(1 + \frac{2}{2 \cdot 625}\right) = 25.040.$$

This result is very accurate, since $\sqrt[3]{627}$ to three correct places is in fact 25.040.

We can show similarly that, in general, $\sqrt{a^2+b}$ can be put approximately equal to $a + \frac{b}{2a}$, if b is small compared with a^2 .

III. We now consider an extremely important limit—the limit of the ratio of the sine to its argument when the argument tends to zero in an arbitrary manner:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x}.$$

We shall seek this limit by starting from the geometrical definition of sine. We take a circle of radius 1; then $|\alpha|$ will measure the length of arc. Since $\frac{\sin \alpha}{\alpha}$ is an even function, it is sufficient to consider the case when $\alpha > 0$. We now proceed as in Sec. 38. We mark off on both sides of any point A (Fig. 51) equal arcs AC and AC' of length $\alpha < \frac{1}{2}\pi$ (by hypothesis, α is made as close as desired to zero, so that we are justified in assuming that α is less than $\frac{1}{2}\pi$); we join points C and C' by a straight line and draw tangents at these points, CD and $C'D$, to their intersection at point D . We have:

$$CB = C'B = \sin \alpha, CD = C'D = \tan \alpha.$$

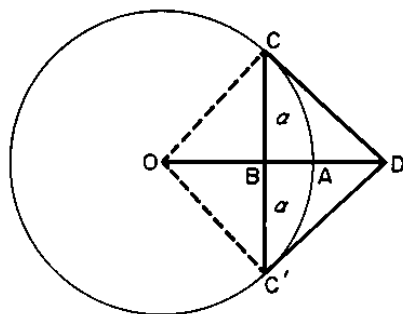


FIG. 51

But the length of the step-line CDC' is greater than the length of arc CAC' , which is in turn greater than the length of the straight segment CC' . Thus

$$2 \sin \alpha < 2\alpha < 2 \tan \alpha.$$

We obtain by dividing through by $2 \sin \alpha$:

$$1 < \frac{\alpha}{\sin \alpha} < \frac{1}{\cos \alpha},$$

or

$$1 > \frac{\sin \alpha}{\alpha} > \cos \alpha.$$

Since $\cos \alpha$ is a continuous function, equal to unity when $\alpha = 0$, as $\alpha \rightarrow 0$ the function $\frac{\sin \alpha}{\alpha}$ lies between functions having the same limit equal to unity. By test 1, Sec. 34, we conclude that

$$\lim_{\alpha \rightarrow 0} \frac{\sin \alpha}{\alpha} = 1.$$

Hence $\sin \alpha$ and α are equivalent infinitesimals: $\sin \alpha \sim \alpha$. With sufficiently small α we can replace $\sin \alpha$ by α with as small a relative error as desired.

The function $y = \frac{\sin x}{x}$ has a removable discontinuity at $x = 0$.

The limits of many other ratios of infinitesimals may be evaluated with the aid of the above limit.

$$(1) \lim_{\alpha \rightarrow 0} \frac{\tan \alpha}{\alpha} = \lim_{\alpha \rightarrow 0} \frac{\sin \alpha}{\alpha} \cdot \frac{1}{\cos \alpha} = \lim_{\alpha \rightarrow 0} \frac{\sin \alpha}{\alpha} \cdot \lim_{\alpha \rightarrow 0} \frac{1}{\cos \alpha} = 1;$$

$$(2) \lim_{\alpha \rightarrow 0} \frac{\sin k\alpha}{\alpha} = \lim_{\alpha \rightarrow 0} \frac{k \sin k\alpha}{k\alpha} = k \lim_{\alpha \rightarrow 0} \frac{\sin k\alpha}{k\alpha} = k \cdot 1 = k;$$

$$(3) \lim_{\alpha \rightarrow 0} \frac{1 - \cos \alpha}{\alpha} = \lim_{\alpha \rightarrow 0} \frac{2 \sin^2 \frac{\alpha}{2}}{\alpha} = \lim_{\alpha \rightarrow 0} \frac{\sin \frac{\alpha}{2}}{\frac{\alpha}{2}} \cdot \sin \frac{\alpha}{2} = 1 \cdot 0 = 0.$$

The last example shows that $1 - \cos \alpha$ is an infinitesimal of higher order than α and $\sin \alpha$ as $\alpha \rightarrow 0$.

41. The number e . Natural logarithms.

I. An important role is played in mathematical analysis by another limit, viz.

$$\lim_{z \rightarrow \infty} \left(1 + \frac{1}{z}\right)^z.$$

As we shall show, this limit exists as z increases indefinitely according to any law. First of all, we introduce the following important definition.

Definition. The number e^* is defined as the limit of the function $\left(1 + \frac{1}{n}\right)^n$ as $n \rightarrow \infty$:

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n.$$

* This notation, like the notation π for the ratio of the circumference of a circle to its diameter, was introduced by EULER.

We shall justify this definition by showing that the limit in fact exists.

We have by Newton's formula:

$$\begin{aligned} u_n &= \left(1 + \frac{1}{n}\right)^n = 1 + \frac{n}{1!} \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \\ &\quad \cdot \frac{1}{n^3} + \dots + \frac{n(n-1)\dots(n-k+1)}{k!} \cdot \frac{1}{n^k} + \dots \\ &\quad \dots + \frac{n(n-1)\dots(n-n+1)}{n!} \frac{1}{n^n} = 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \\ &\quad + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \dots + \frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \\ &\quad \dots \left(1 - \frac{k-1}{n}\right) + \dots + \frac{1}{n!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{n-1}{n}\right). \end{aligned}$$

When n increases, each term (except the first two) in the last expression for u_n increases: the general term of the sum:

$$\frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{k-1}{n}\right)$$

becomes greater as n becomes larger; in addition, new positive terms are added. Hence u_n is an increasing function of n . But it is bounded; for, on replacing the fractions in brackets in all the terms by unities, we obtain:

$$u_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} = s_n < 3 \quad (*)$$

(see the example at the end of Sec. 34). Thus u_n is a bounded monotonic function, and tends to a limit, which evidently lies between 2 and 3*.

* We remark that $s_n = 1 + 1 + 1/2! + 1/3! + \dots + 1/n!$ tends to the same limit: $\lim s_n = e$. Let us prove this. We wrote s in Sec. 34 for the limit of s_n . We want to show that $s = e$. If we stop short at the $(k+1)$ -th term of the expression for u_n ($k < n$), we get:

$$\begin{aligned} u_n &> 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \dots \\ &\quad \dots + \frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{k-1}{n}\right). \end{aligned}$$

When $n \rightarrow \infty$ (with k constant), we arrive at the relationship

$$e > 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{k!} = s_k. \quad (**)$$

The number e is irrational, and hence cannot be expressed by any finite fraction. We shall give its value correct to fifteen places after the decimal:

$$e = 2.718281828459045 \dots$$

We generally confine ourselves to the first two or three places in practical work.

THEOREM. The function $\left(1 + \frac{1}{z}\right)^z$ has a limit equal to e when $z \rightarrow \infty$ in an arbitrary manner:

$$\lim_{z \rightarrow \infty} \left(1 + \frac{1}{z}\right)^z = e.$$

Proof. Let z tend to $+\infty$ in an arbitrary manner. Every value of it evidently lies between two successive integers n and $n+1$, $n \leq z \leq n+1$. It is clear that

$$\left(1 + \frac{1}{n+1}\right)^n \leq \left(1 + \frac{1}{z}\right)^z \leq \left(1 + \frac{1}{n}\right)^{n+1}$$

or

$$u_{n+1} \frac{1}{1 + \frac{1}{n+1}} \leq \left(1 + \frac{1}{z}\right)^z \leq u_n \left(1 + \frac{1}{n}\right),$$

where, as before, $u_n = \left(1 + \frac{1}{n}\right)^n$. But if $z \rightarrow +\infty$, $n \rightarrow \infty$ also, and the functions, between which the function $\left(1 + \frac{1}{z}\right)^z$ lies, now both tend to the number e :

$$\lim_{n \rightarrow \infty} \left(u_{n+1} \frac{1}{1 + \frac{1}{n+1}} \right) = \lim_{n \rightarrow \infty} u_{n+1} \cdot \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n+1}} = e \cdot 1 = e,$$

$$\lim_{n \rightarrow \infty} \left[u_n \left(1 + \frac{1}{n}\right) \right] = \lim_{n \rightarrow \infty} u_n \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = e \cdot 1 = e.$$

Continued from p. 125:

Hence, we have, by inequalities (*) and (**):

$$u_k < s_k < e.$$

Since now, $u_k \rightarrow e$ as $k \rightarrow \infty$, it follows from this that

$$\lim_{k \rightarrow \infty} s_k = s = e.$$

This is what we wanted to prove.

It follows that $\left(1 + \frac{1}{z}\right)^z$ tends to the number e , whatever the manner in which the variable z tends to positive infinity.

It remains for us to consider the case when z tends to $-\infty$ in an arbitrary manner. We put $z = -(t+1)$; then $t \rightarrow +\infty$ as $z \rightarrow -\infty$. We have

$$\begin{aligned}\lim_{z \rightarrow -\infty} \left(1 + \frac{1}{z}\right)^z &= \lim_{t \rightarrow +\infty} \left(1 - \frac{1}{t+1}\right)^{-(t+1)} = \lim_{t \rightarrow +\infty} \left(\frac{t}{t+1}\right)^{-(t+1)} \\ &= \lim_{t \rightarrow +\infty} \left(1 + \frac{1}{t}\right)^{t+1} = \lim_{t \rightarrow +\infty} \left(1 + \frac{1}{t}\right)^t \cdot \lim_{t \rightarrow +\infty} \left(1 + \frac{1}{t}\right) = e \cdot 1 = e;\end{aligned}$$

thus our assertion is fully proved.

We could therefore define e as the limit of the function $\left(1 + \frac{1}{z}\right)^z$ as $z \rightarrow \infty$.

II. If $z = 1/x$, $x \rightarrow 0$ as $z \rightarrow \infty$. Therefore,

$$\lim_{z \rightarrow \infty} \left(1 + \frac{1}{z}\right)^z = \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e.$$

We now use this limit relationship to consider a further interesting ratio of two infinitesimals.

Let us find

$$\lim_{x \rightarrow 0} \frac{\log_a(1+x)}{x}, \quad a > 0.$$

We are in fact concerned here with a ratio of infinitesimals, since $\lim_{x \rightarrow 0} \log_a(1+x) = \log_a 1 = 0$.

$x \rightarrow 0$

We carry out the following transformations:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\log_a(1+x)}{x} &= \lim_{x \rightarrow 0} \left[\frac{1}{x} \log_a(1+x) \right] = \lim_{x \rightarrow 0} \log_a(1+x)^{\frac{1}{x}} \\ &= \log_a \left[\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} \right];\end{aligned}$$

the last is possible because of the continuity of the logarithm. We find from the limit relationship just established:

$$\lim_{x \rightarrow 0} \frac{\log_a(1+x)}{x} = \log_a e.$$

This shows that $\log_a(1+x)$ and x are infinitesimals of the same order as $x \rightarrow 0$. It is quite obvious that, if the number e is taken as the base a of the logarithmic function, then $\log_e e = 1$, so that $\log_e(1+x)$ and x are equivalent infinitesimals as $x \rightarrow 0$:

$\log_e(1+x) \sim x$. With sufficiently small x , we can replace $\log_e(1+x)$ by x with as small a relative error as desired. Without question, this must greatly assist in simplifying working, and e is in fact the most convenient base in analysis for the logarithmic function. Logarithms to base e are denoted by the symbol $\ln$ and are called *natural* or *Naperian* logarithms, after the name of one of the first inventors of logarithmic tables—Napier (1550–1617).

Since the logarithms of a number to two different bases are proportional to one another (Sec. 23), there is no difficulty about passing from natural logarithms to those of any other system, say to base ten, or vice versa.

Natural and common logarithms are connected by the formulae:

$$\log x = M \ln x \quad \text{and} \quad \ln x = \frac{1}{M} \log x$$

where

$$M = \log e = \frac{1}{\ln 10} = 0.434294 \left(\frac{1}{M} = 2.302585 \right).$$

The number M is called the transition modulus from natural to common logarithms. Common logarithms are always used in practical calculations; whilst they have no theoretical advantages over any other system, they are more convenient in computations because our system of arithmetic is itself based on ten.

We shall always use the natural system of logarithms in this book.

Since the functions $y = (1+x)^{1/x}$ and $y = \frac{\ln(1+x)}{x}$ have limits (respectively e and 1) as $x \rightarrow 0$, $x = 0$ is a point of removable discontinuity of these functions.

The limits of many other functions may be found with the aid of the above limits.

Examples. (1) We have on putting $x = kz$:

$$\begin{aligned} (1) \quad \lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^x &= \lim_{z \rightarrow \infty} \left(1 + \frac{1}{z}\right)^{zk} = \lim_{z \rightarrow \infty} \left[\left(1 + \frac{1}{z}\right)^z\right]^k \\ &= \left[\lim_{z \rightarrow \infty} \left(1 + \frac{1}{z}\right)^z\right]^k = e^k. \end{aligned}$$

(2) Putting $k\alpha = z$, we have:

$$\lim_{\alpha \rightarrow 0} \frac{\ln(1+k\alpha)}{\alpha} = \lim_{z \rightarrow 0} k \frac{\ln(1+z)}{z} = k \lim_{z \rightarrow 0} \frac{\ln(1+z)}{z} = k \cdot 1 = k.$$

(3) Putting $a^h - 1 = z$, we have: $a^h = 1 + z$, $h = \frac{\ln(1+z)}{\ln a}$, i.e.

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = \lim_{z \rightarrow 0} \frac{z}{\frac{\ln(1+z)}{\ln a}} = \lim_{z \rightarrow 0} \left[\ln a \frac{z}{\ln(1+z)} \right] = \ln a.$$

We shall need these examples in what follows.

DERIVATIVES AND DIFFERENTIALS. THE DIFFERENTIAL CALCULUS

1. The Concept of Derivative. Rate of Change of a Function

42. Some physical concepts. Let us consider the following simple physical phenomena: (1) rectilinear motion; (2) linear expansion of a mass; (3) heating of a body. To characterize these phenomena, the respective concepts have been introduced: (1) velocity, (2) density, (3) specific heat; as it happens, all these represent particular aspects of the same concept from the mathematical point of view. We shall show this by analyzing each phenomenon separately.

I. VELOCITY OF RECTILINEAR MOTION. Suppose that a body performs a rectilinear motion, and that we know the distance s traversed by the body after any given time t , i.e. we know s as a function of t :

$$s = F(t)$$

The equation $s = F(t)$ is called the equation of motion, whilst the curve defined by it in the system of axes Ots is the graph of the motion.

Let us consider the motion of the body in the course of an interval of time Δt from the instant t to the instant $t + \Delta t$. The body has traversed a distance $s = F(t)$ after time t , and $s + \Delta s = F(t + \Delta t)$ after time $t + \Delta t$. Thus in Δt units of time it travels

$$\Delta s = F(t + \Delta t) - F(t).$$

If the motion is uniform, s is a linear function of t :

$$s = v_0 t + s_0$$

(Sec. 17). In this case $\Delta s = v_0 \Delta t$, and the ratio $\Delta s / \Delta t (= v_0)$ shows how many units of path s are traversed in unit time t ; the ratio remains constant here, i.e. depends neither on the instant t chosen, nor on

the increment of time Δt . This constant ratio $\frac{\Delta s}{\Delta t}$ is called the velocity of the uniform motion*. If the motion is non-uniform, the ratio $\Delta s/\Delta t$ of path to time depends both on t and on Δt . It is called the average velocity of the motion in the time interval from t to $t + \Delta t$ and is written as v_{av} :

$$v_{av} = \Delta s / \Delta t.$$

In this time interval, the motion may be of entirely different types for the same distance traversed; this is illustrated graphically by the fact that we can draw entirely different curves (AC_1B , AC_2B , AC_3B) between two points (A and B in Fig. 52) of the Ots plane; these curves are the graphs of quite different motions in the given time intervals, all the motions having the same average velocity $\Delta s/\Delta t$. In particular, we can draw between points A and B the straight segment ACB , this being the graph of a motion which is uniform in the interval $(t, t + \Delta t)$. Thus the average velocity $\Delta s/\Delta t$ shows the uniform velocity at which we need to travel in order to traverse the distance Δs in the time interval $(t, t + \Delta t)$.

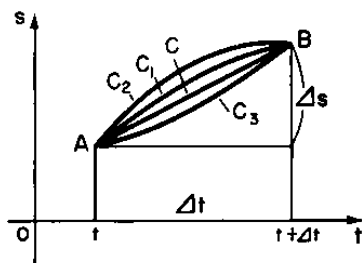


FIG. 52

Let Δt diminish, whilst t remains as before.

The average velocity, calculated for the new interval $(t, t + \Delta t_1)$, $\Delta t_1 < \Delta t$, lying inside the given interval, will in general differ from that for the whole interval $(t, t + \Delta t)$. It follows that we cannot regard average velocity as a sufficient characteristic for the motion, inasmuch as it depends on the interval for which the calculation is made. If we take as our starting-point the fact that the average velocity in the interval $(t, t + \Delta t)$ must be regarded as the better characterizing the motion, the smaller Δt , we must let Δt tend to zero. If v_{av} now has a limit, this is in fact taken as the velocity at the

* Numerically, the velocity of a uniform motion is equal to the distance traversed in unit time; velocity has dimensions cm/sec in the CGS system.

given instant t . This magnitude does not depend on the method of calculating it.

The velocity v of a rectilinear motion at a given instant t (at a given point t) is defined as the limit (if it exists) of the average velocity v_{av} corresponding to the interval $(t, t + \Delta t)$ as Δt tends to zero:

$$v = \lim_{\Delta t \rightarrow 0} v_{av} = \lim_{\Delta t \rightarrow 0} \Delta s / \Delta t = \lim_{\Delta t \rightarrow 0} \frac{F(t + \Delta t) - F(t)}{\Delta t}.$$

Example. We again take the law of free fall (Sec. 9):

$$s = \frac{1}{2} g t^2.$$

We have for the average velocity of the fall in the time interval $(t, t + \Delta t)$:

$$v_{av} = \frac{\frac{1}{2} g (t + \Delta t)^2 - \frac{1}{2} g t^2}{\Delta t} = \frac{1}{2} g \frac{2t \Delta t + (\Delta t)^2}{\Delta t} = \frac{1}{2} g (2t + \Delta t),$$

and for the velocity at the instant t :

$$v = \lim_{\Delta t \rightarrow 0} \frac{1}{2} g (2t + \Delta t) = g t.$$

It is clear from this that the velocity is completely defined by the instant t ; it is proportional here to the time of the motion (of the fall).

II. DENSITY. We now turn to an important concept from another domain of physics, defined in precisely the same way as velocity.

We take a material curve (say a wire) and move from one end to the other, meantime measuring the length s of the piece traversed and also its mass m . There is a corresponding definite mass m for each value of s , so that the former is a function of s :

$$m = \Phi(s).$$

We say that matter is distributed uniformly along the curve, or that the material curve is homogeneous, if any two pieces of equal length contain equal masses. In this case m is a linear function of s , or more precisely: m is directly proportional to s :

$$m = \delta_0 s,$$

where δ_0 is a constant coefficient of proportionality, whilst $\Delta m = \delta_0 \Delta s$, and the ratio $\Delta m / \Delta s (= \delta_0)$ shows how many units of mass m there are per unit length s . The constant ratio is called the (linear) density of the homogeneous curve.

Let the material be distributed non-uniformly. We take the piece of curve for which the length changes from s to $s + \Delta s$; the mass Δm of this piece is given by

$$\Delta m = \Phi(s + \Delta s) - \Phi(s).$$

The ratio $\Delta m/\Delta s$ of mass to length is no longer constant, but depends on both s and Δs ; it is called the average linear density δ_{av} of the curve (wire):

$$\delta_{av} = \Delta m/\Delta s.$$

Given the same mass Δm , the material can be distributed in very different ways (and in particular uniformly) in the interval $(s, s + \Delta s)$. Thus the average density $\Delta m/\Delta s$ indicates with what density the material needs to be uniformly distributed in the interval $(s, s + \Delta s)$ in order that its total mass remains the same, Δm . The average density $\Delta m/\Delta s$ merely characterizes "on the whole" or "on the average" the distribution of the material in the given piece. In order to avoid this indeterminacy, the concept of density at a point of the curve (at a given s) is introduced. This concept is established by starting from the idea that the average densities corresponding to a decreasing length Δs provide an increasingly better characteristic for the material distribution.

The linear density δ of a material curve at a given point s is defined as the limit (if it exists) of the average linear density δ_{av} corresponding to the interval $(s, s + \Delta s)$ as Δs tends to zero:*

$$\delta = \lim_{\Delta s \rightarrow 0} \delta_{av} = \lim_{\Delta s \rightarrow 0} \Delta m/\Delta s = \lim_{\Delta s \rightarrow 0} \frac{\Phi(s + \Delta s) - \Phi(s)}{\Delta s}.$$

III. SPECIFIC HEAT. Let us take a further example from heat — that of specific heat.

If the temperature of a body of mass 1 g is increased, say from 0 to τ , this occurs as a result of a definite quantity of heat q being transmitted to the body; hence q is a function of the temperature τ to which the body is raised:

$$q = \Psi(\tau)$$

* The density of uniformly distributed material is numerically equal to the mass contained in unit length of the curve; the dimensions of linear density in the CGS system are g/cm.

Let the temperature of the body be raised by $\Delta\tau$. The amount of heat Δq expended in doing this is given by

$$\Delta q = \Psi(\tau + \Delta\tau) - \Psi(\tau).$$

If the amount of heat q were proportional to the temperature (i.e. if q varies uniformly with variation of τ), we have $\Delta q = c_0 \Delta\tau$, and the constant ratio $\Delta q / \Delta\tau (= c_0)$ would indicate how many units of heat q there were per unit of temperature τ . Experiments show, however, that $\Delta q / \Delta\tau$ is not constant, but depends on τ and $\Delta\tau$; it gives us the amount of heat which is needed "on the average" for heating the body 1° in the considered temperature interval $(\tau, \tau + \Delta\tau)$. This ratio is called the average specific heat c_{av} of the body in the interval $(\tau, \tau + \Delta\tau)$:

$$c_{av} = \Delta q / \Delta\tau.$$

Since there is an indeterminacy implicit in the concept of average specific heat, just as in the concepts of average velocity and average density, we introduce as in the previous cases the concept of specific heat at a given temperature τ .

The specific heat c at a temperature τ (at a given point τ) is defined as the limit (if it exists) of the average specific heat corresponding to the interval $(\tau, \tau + \Delta\tau)$ as $\Delta\tau$ tends to zero:*

$$c = \lim_{\Delta\tau \rightarrow 0} c_{av} = \lim_{\Delta\tau \rightarrow 0} \Delta q / \Delta\tau = \lim_{\Delta\tau \rightarrow 0} \frac{\Psi(\tau + \Delta\tau) - \Psi(\tau)}{\Delta\tau}.$$

IV. THE RATE OF CHANGE OF A FUNCTION. A special concept has been defined in each of the above examples, yet in spite of all the differences in the physical nature of the concepts, they represent from the mathematical point of view the same characteristic of the corresponding function.

We now analyse the question in the general form as applied to a function $y = f(x)$, paying no attention to the physical significance of x and y .

First of all, let $f(x)$ be a linear function: $y = f(x) = ax + b$. If the independent variable x receives an increment Δx , the function y now receives an increment $\Delta y = a\Delta x$. The ratio $\Delta y / \Delta x (= a)$ indicates how many units of y there are per unit of x , and remains constant independently of Δx or of the x taken.

* Specific heat is measured in calories; the calorie has the dimension of energy, i.e. in the CGS system, $\text{g}/\text{cm}^2/\text{sec}^2$.

Definition. The constant ratio $\Delta y/\Delta x = a$ is called the rate of change of the linear function $y = ax + b$.

If the function $y = f(x)$ is not linear, then

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

depends both on x and Δx . But this ratio characterizes the function "on the average" (in the usual sense) when the independent variable changes from x to $x + \Delta x$; it is the rate of change of the linear function which has the same increment Δy in the same interval $(x, x + \Delta x)$.

Definition. The ratio $\Delta y/\Delta x$ is called the **average rate of change** v_{av} of the function in the interval $(x, x + \Delta x)$.

We now reflect that the average rate of change characterizes the function better and better as the length of the interval $|\Delta x|$ diminishes (i.e. as we get closer and closer to the point x). This reflection is based on the following obvious fact: different graphs of functions $y = f(x)$, passing through two points $A(x, y)$ and $B(x + \Delta x, y + \Delta y)$ (see Fig. 52), differ less and less, in general, between these points from the straight line AB , the smaller AB becomes. In view of this, we let Δx tend to zero and consider the behaviour in the limit of the average rate of change of the function. If the limit exists, it is taken as a measure of the "speed" or "rate" of change of the function for the given x and is called the rate of change of the function.

Definition. The **rate of change** v of the function $y = f(x)$ for a given x (at a given point x) is defined as the limit (if it exists) of the average rate of change $v_{av} = \Delta y/\Delta x$ of the function in the interval $(x, x + \Delta x)$ as Δx tends to zero:

$$v = \lim_{\Delta x \rightarrow 0} v_{av} = \lim_{\Delta x \rightarrow 0} \Delta y/\Delta x = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Using the new terminology, we can say that:

- (1) velocity is the rate of change of distance as a function of time;
- (2) linear density is the rate of change of mass as a function of length;
- (3) specific heat is the rate of change of the amount of heat as a function of temperature.

The concept of the rate of change of a function at a point is of the greatest value both in mathematics and in other sciences, where it takes a concrete form in fundamental and important special concepts.

43. Derivative of a function. The rate of change of a function $y = f(x)$ is determined by means of the following consecutive operations:

(1) the increment of the function is found, corresponding to the increment Δx assigned to a given value $x = x_0$:

$$\Delta y = f(x_0 + \Delta x) - f(x_0);$$

(2) the ratio $\Delta y / \Delta x$ is formed;

(3) the limit (if it exists) of this ratio is found as Δx tends to zero in an arbitrary manner.

This limit is generally called the derivative of the function $f(x)$ at $x = x_0$; it is written as $f'(x_0)$, which is read as " f dash x_0 ".

Definition. The *numerical derivative* of a function is defined as the limit of the ratio of the increment of the function to the increment of the given value of the independent variable as the latter increment tends to zero in an arbitrary manner.

Suppose that the numerical derivative of a function $f(x)$ exists for every value x of the independent variable in an interval; we now have a corresponding number for every point of the interval, i.e. a new function; this new function is called the derivative of $f(x)$ and is written as $f'(x)$.

The derivative of a given function is found with the aid of the same operations as for the numerical derivative, except that the independent variable has arbitrary values in an interval.

Definition. The *derivative* of a function is defined as the limit of the ratio of the increment of the function to the increment of the independent variable when the latter increment tends arbitrarily to zero:

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

The numerical derivative is a particular value of the derivative, for the value concerned of the independent variable.

We can therefore say that the rate of change of a function $y = f(x)$ is expressed by the derivative of y with respect to x and in particular, that (1) the velocity of a rectilinear motion is the derivative

of the distance with respect to time; (2) the linear density is the derivative of the mass with respect to length; (3) the specific heat is the derivative of the amount of heat with respect to temperature.

The derivative represents a fundamental concept of mathematical analysis. I. Newton*, one of the founders of analysis, arrived at the idea of derivative, then at the construction of the differential calculus, by starting from the question of the velocity of a moving point.

Let us find the derivatives of some elementary functions. Let $y = x$. We have:

$$y' = (x)' = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} = 1,$$

i.e. the derivative $(x)'$ is a constant equal to 1. This is obvious, since $y = x$ is a linear function and its rate of change is constant.

If $y = x^2$, then

$$y' = (x^2)' = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = 2x$$

(cf. the examples in Sec. 42, 1).

Let $y = x^3$, then

$$y' = (x^3)' = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x} = 3x^2.$$

The regularity may easily be observed in the expression for the derivatives of the power function $y = x^n$ with $n = 1, 2, 3$. We shall show that, in general, the derivative of $y = x^n$ is nx^{n-1} for any positive integral exponent n .

We have:

$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x}.$$

The numerator can be transformed by using Newton's formula for a binomial with an integral power, or another familiar algebraic formula can be used:

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1}).$$

* I. NEWTON (1642-1727)—a great English scholar—was a mathematician, physicist and astronomer. Newton laid the foundations of many branches of modern science and completed the creation of the bases of mathematical analysis. His work helped to overthrow the mediaeval scholastics and was an invaluable asset in developing a genuinely scientific materialistic world-picture.

We get with the aid of this latter:

$$\frac{\Delta y}{\Delta x} = \frac{[(x + \Delta x) - x][(x + \Delta x)^{n-1} + (x + \Delta x)^{n-2}x + \cdots + (x + \Delta x)x^{n-2} + x^{n-1}]}{\Delta x}$$

i.e.

$$\frac{\Delta y}{\Delta x} = (x + \Delta x)^{n-1} + (x + \Delta x)^{n-2}x + \cdots + (x + \Delta x)x^{n-2} + x^{n-1}.$$

The right-hand side is the sum of n terms, each of which tends to x^{n-1} as $\Delta x \rightarrow 0$. Thus

$$(x^n)' = nx^{n-1}.$$

This result should be memorized; as we shall show later (Sec. 48), it remains true, not only for a positive integer n , but for any real n .

For instance,

$$(\sqrt[3]{x})' = (x^{\frac{1}{3}})' = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}.$$

This general formula, with $n = 1, 2, 3$, leads to the formulae found above by other means.

We consider separately the derivative of a constant:

$$y = C.$$

Since this function does not change when the independent variable changes, we have $\Delta y = 0$ and $\Delta y/\Delta x = 0/\Delta x = 0$. Consequently,

$$(C)' = 0,$$

i.e. *the derivative of a constant is zero*. This is in complete accord with our direct consideration of "rate of change", which must be zero for a constant because it undergoes no sort of change.

44. Geometrical interpretation of derivative. The derivative of a function $y = f(x)$ permits an extremely simple visual interpretation, connected with the concept of tangent line to a curve*.

* Unless there is a special proviso, we shall always think of a function $y = f(x)$ as represented geometrically by a graph in a Cartesian co-ordinate system Oxy . For the geometrical meaning of the derivative when the graph is drawn in a system of polar co-ordinates, see Sec. 56.

Definition. The *tangent** MT to a curve at a point M (Fig. 53) is defined as the straight line passing through M and occupying the limiting position of the secant through M and another point M' of the curve, when M' tends to merge with the given point M .

The "limiting position" of the straight line is defined by the fact that the angle $TM M'$ tends to zero along the chord MM' .

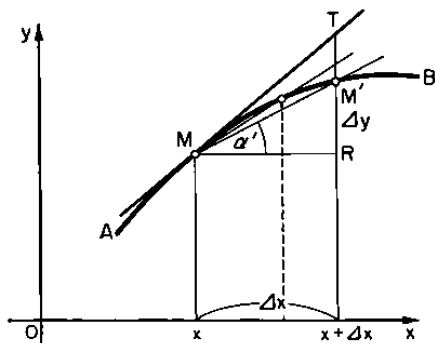


FIG. 53

The geometrical meaning of the derivative follows from the next proposition.

THEOREM. The value of the derivative $f'(x)$ (the numerical derivative) is equal to the tangent of the angle formed with Ox by the tangent to the graph of $y = f(x)$ at the point with abscissa x ; more briefly:

The value of the derivative $f'(x)$ is equal to the slope of the tangent to the graph of $y = f(x)$ at the point with abscissa x .

Proof. Let Δx be the increment assigned to any given x ; for the two values x and $x + \Delta x$ there are two corresponding points M and M' on the curve AB —the graph of the function $y = f(x)$ (Fig. 53). The increment of the function $\Delta y = f(x + \Delta x) - f(x)$ is represented by the segment RM' , and consequently, the ratio $\Delta y / \Delta x = M'R / MR$ is $\tan \angle M'MR$. In accordance with the definition of derivative, we indefinitely decrease the increment Δx ; the point M' now indefinitely approaches M along the curve AB and tends to merge with it, whilst the secant MM' , rotating about the point M , tends to the tangent MT . Thus the left-hand side of the equation

$$\tan \angle M'MR = \frac{\Delta y}{\Delta x}$$

* We usually speak simply of "tangent" rather than "tangent line".

will tend to $\tan \angle TMR$ as $\Delta x \rightarrow 0$, and the right-hand side to $f'(x)$; we get:

$$\tan \angle TMR = f'(x).$$

If we can draw the tangent to the graph of a function, we can determine the corresponding value of the derivative.

In fact, however, we do not find the derivative of a function from its tangent; on the contrary, we are enabled to draw the tangent to the graph from our knowledge of the derivative.

It may be mentioned here that the problem of accurately defining and constructing the tangent to a curve led Leibniz* (at the same time as Newton) to the concept of derivative and the creation of the differential calculus.

The analytic methods described may be used to solve a variety of geometrical problems on tangents and related concepts, of which we shall consider the following: the direction of a curve, the angle between two curves, the normal to a curve, and the subtangent and subnormal.

Definition 1. The direction of a curve at a point M_0 is defined as the direction of the tangent to the curve at M_0 (Fig. 54); the direction of a straight line can be characterized by its angle of inclination α to Ox , or what amounts to the same thing, by its slope $\tan \alpha$, i.e. in our case, by the derivative of the function at the point x_0 —the abscissa of the point M_0 :

$$f'(x_0) = \tan \alpha.$$

Definition 2. The angle between two intersecting curves is the angle between the tangents to the curves at their point of intersection.

Definition 3. The normal to a curve at a point M_0 is defined as the straight line M_0N (Fig. 54) passing through M_0 and perpendicular to the tangent at this point.

Definition 4. The subtangent of a curve at a point M_0 is defined as the projection T_0P_0 on Ox of the directed segment T_0M_0 of the tangent to the curve at the point M_0 (Fig. 54).

Definition 5. The subnormal of a curve at a point M_0 is defined as the projection N_0P_0 on Ox of the directed segment N_0M_0 of the normal to the curve at M_0 (Fig. 45).

* G. Leibniz (1646–1716) was a great German mathematician. Along with Newton, he laid the foundations of mathematical analysis. He was responsible for a large number of original discoveries in the differential and integral calculus, and in particular, for a system of notation which proved more convenient than that of Newton.

Let us form the equations of the tangent and normal to the curve $y = f(x)$ at the point $M_0(x_0, y_0)$ and find expressions for the subtangent and subnormal.

The equation of a straight line through the point $M_0(x_0, y_0)$ is

$$y - y_0 = k(x - x_0), \quad \text{where} \quad y_0 = f(x_0).$$

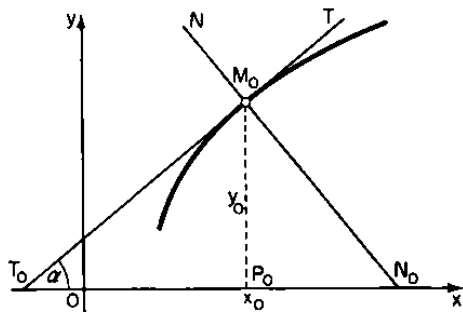


FIG. 54

On putting $k = f'(x_0)$ in this equation, we obtain the required equation of the tangent:

$$y - y_0 = f'(x_0)(x - x_0).$$

Since the normal is perpendicular to the tangent, we obtain the equation of the normal by putting $k = -1/f'(x_0)$:

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0).$$

We now find the abscissa of the point of intersection of the tangent with Ox . We put $y = 0$ in the equation of the tangent; this gives $-y_0 = f'(x_0)(x - x_0)$, whence

$$x = x_0 - \frac{y_0}{f'(x_0)}.$$

The subtangent T_0P_0 is contained between this point and the point $x = x_0$, i.e. is equal to $x_0 - \left(x_0 - \frac{y_0}{f'(x_0)}\right)$, i.e.

$$T_0P_0 = \frac{y_0}{f'(x_0)}.$$

The expression for the subnormal N_0P_0 is similarly obtained:

$$N_0P_0 = -y_0 f'(x_0).$$

45 Some properties of the parabola. As an example of the application of the derivative to geometrical problems, we consider the elementary properties of the tangents to a parabola.

Let $y = ax^2$ be the parabola (Fig. 55). As is easily verified, $(ax^2)' = 2ax$, so that the subtangent SN corresponding to the point $M(x_0, y_0)$ is equal to $y_0/2ax_0$; but $y_0 = ax_0^2$, so that $SN = ax_0^2/2ax_0 = \frac{1}{2}x_0$, i.e. the subtangent of the parabola is equal to half the abscissa of the point of contact.

This fact alone is sufficient for us to deduce several important properties of the parabola by purely geometrical means.

We notice first of all that the congruence of triangles SOT and SMN (Fig. 55) implies that

$$OT = MN \text{ and } ST = SM$$

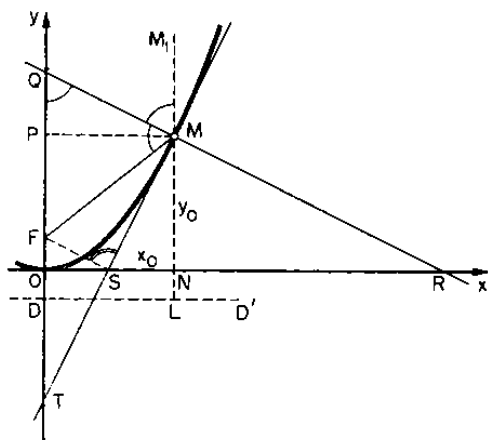


FIG. 55

I. To draw the tangent to a parabola at a point M , we project this point on to the parabola axis and take the point T on the continuation of the axis through the vertex O at a distance $OT = OP$. The straight line joining points T and M is in fact the tangent required. We can also project the point M on to the tangent at the vertex and take the point S bisecting segment ON .

II. We draw the straight line joining point S to the focus F of the parabola. Since $MF = ML$ (DD' is the directrix), whilst

$$ML = MN + NL = OT + OF = FT,$$

the triangle TFM is isosceles, i.e. FS is perpendicular to the tangent MT .

This provides us with a method of finding the focus. Conversely, if we know the axis, vertex and focus, we can use this property to construct the parabola itself. For let OO' be the parabola axis (Fig. 56), O the vertex and F the focus. We erect the perpendicular OR to OO' at O . If we draw a straight line from F cutting OR in some point S , the perpendicular to FS at S will be a tangent to the parabola. If we draw from F an entire pencil of straight lines cutting OR , we can find the same number of straight lines tangential to the parabola. The curve outlined by this family of straight lines is in fact the required parabola. The family is said to form the envelope of the curve in such cases.

The more tangents we draw, the more accurately the parabola can be traced. This method provides a construction of the parabola from its tangents instead of its points.

III. We now return to Fig. 55 and draw the normal MQ to the parabola at the point M . Obviously, FS is the centre line of the right-angled triangle QMT . Therefore $FQ = FT$, whence $FQ = FM$. Thus $\angle FQM = \angle QMF$. If we produce the ordinate of M , we have

$$\angle QMF = \angle QMM_1.$$

Thus the normal at any point of a parabola bisects the angle between the focal radius and the straight line parallel to the axis through this point.

This property of a parabola is widely used in technology. For instance, a projector or automobile lamp is usually given the shape of a paraboloid of revolution, i.e. the surface obtained by rotating a parabola about its axis. This shape is chosen for the following reason. If a light-source is arranged at the focus of the parabola, a ray incident on the smoothly polished surface of the reflector remains, after reflection, in the same plane as the incident ray and the normal to the surface, as follows from the familiar law of physics. In accordance with the same law, the angle of incidence must be equal to the angle of reflection, which implies, by virtue of the property just proved for a parabola, that the reflected ray is parallel to the axis of the reflector. Hence all the rays travel in a parallel beam after reflection, which naturally gives the strongest illumination at great distances.

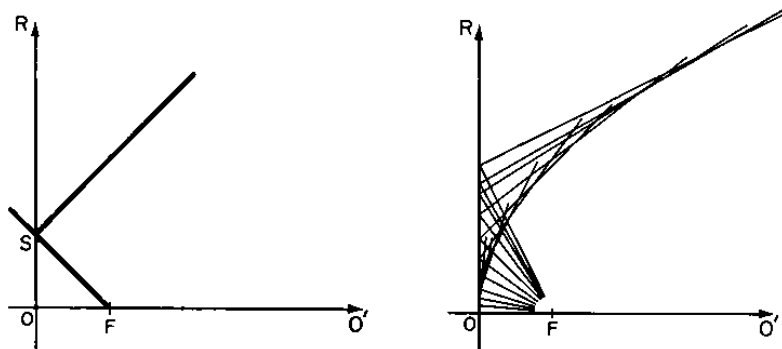


FIG. 56

Similarly, various arrangements for the best propagation or detection of sound are given a parabolic shape (orchestral platforms, sound locators).

2. Differentiation of Functions

46. Differentiation of the results of arithmetic operations. The operation of finding a derivative is called differentiation (an explanation of this name is given a little later, Sec. 51), so that "to differentiate

a function $f(x)$ " means to find its derivative $f'(x)$. To do this we must find

$$\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Such direct differentiation is an extremely laborious and difficult operation in the majority of cases. But if we know, once and for all, the derivatives of all the basic elementary functions, together with the rules for differentiating the results of the arithmetic operations and functions of a function, we can find the derivatives of any elementary functions, without carrying out the above passage to the limit each time. Hence the operation of differentiation can be made automatic for the class of functions of most concern to us.

We turn first of all to the rules for differentiation of a sum, product and quotient. The proofs of the theorems below show in the first place the existence of the derivatives of a sum, product, and quotient, if the derivatives of the components exist (i.e. the terms, factors, divisor and dividend). Naturally, the theorems themselves provide more: they indicate how to form the derivative of a sum, product or quotient from the derivatives of the components.

We shall always assume below that the component functions are continuous and have derivatives (see Sec. 54).

THEOREM I. The derivative of the sum of a finite number of functions is equal to the sum of the derivatives of the terms.

Proof. Let y be given as the sum of functions $u, v, \dots, w$ of the same independent variable x :

$$y = u + v + \dots + w.$$

We show that

$$y' = (u + v + \dots + w)' = u' + v' + \dots + w'.$$

Let the independent variable x be given the increment Δx ; the functions $u, v, \dots, w$ receive increments which we write as $\Delta u, \Delta v, \dots, \Delta w$ respectively, whilst y receives the increment Δy . It is clear that the "incremented" value of the sum will be equal to the sum of the "incremented" values of the terms, i.e.

$$y + \Delta y = (u + \Delta u) + (v + \Delta v) + \dots + (w + \Delta w).$$

On subtracting from this the "original" value of the function, we get

$$\Delta y = \Delta u + \Delta v + \dots + \Delta w,$$

whence we find, on dividing both sides by Δx :

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} + \cdots + \frac{\Delta w}{\Delta x}.$$

On passing to the limit as $\Delta x \rightarrow 0$ and using the theorem on the limit of a sum (Sec. 32), we have:

$$\begin{aligned}\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} + \cdots + \frac{\Delta w}{\Delta x} \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} + \cdots + \lim_{\Delta x \rightarrow 0} \frac{\Delta w}{\Delta x};\end{aligned}$$

since

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = y', \quad \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} = u', \quad \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} = v', \quad \dots, \quad \lim_{\Delta x \rightarrow 0} \frac{\Delta w}{\Delta x} = w',$$

we in fact arrive at the required equation.

THEOREM II. The derivative of the product of two functions is equal to the sum of the first function multiplied by the derivative of the second and the second function multiplied by the derivative of the first.

Proof. Let y be given as the product of two functions:

$$y = uv.$$

We show that

$$y' = (uv)' = uv' + vu'.$$

Let the independent variable receive the increment Δx ; then functions u , v , y receive corresponding increments Δu , Δv , Δy . The "incremented" value of the function y will be

$$y + \Delta y = (u + \Delta u)(v + \Delta v).$$

Hence

$$\begin{aligned}\Delta y &= (y + \Delta y) - y = (u + \Delta u)(v + \Delta v) - uv \\ &= u\Delta v + v\Delta u + \Delta u\Delta v\end{aligned}$$

and

$$\frac{\Delta y}{\Delta x} = u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x}.$$

We find by applying the familiar rules for passage to the limit:

$$\begin{aligned}\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} &= u \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} + v \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \rightarrow 0} \Delta u \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} \\ &= uv' + vu' + 0 \cdot v' = uv' + vu'.\end{aligned}$$

This is what we had to prove.

If one of the factors is a constant, say $v = c$, we get

$$y' = (cu)' = uc' + cu' = cu'$$

since the derivative of a constant is zero. Thus a constant can be taken outside the symbol of the derivative.

The rule for differentiation of the product of two functions can be extended successively to the product of any finite number of functions.

Thus, if $y = uvw$, then

$$y' = (uvw)' = (uv)w' + w(uv)' = uvw' + u w v' + v w u',$$

i.e. the derivative of the product of three functions is equal to the sum of three terms, each of which is the product of two of the functions with the derivative of the third.

Example. Let us find the derivative of the function

$$y = (2x + 3)(1 - x)(x + 2)$$

Knowing that $(x^n)' = nx^{n-1}$ (Sec. 43), we can multiply out the product and differentiate the resulting third degree polynomial. But it is easier to use rule II. We have:

$$\begin{aligned} y' &= (2x + 3)'(1 - x)(x + 2) + (2x + 3)(1 - x)'(x + 2) + \\ &\quad + (2x + 3)(1 - x)(x + 2)' \\ &= 2(1 - x)(x + 2) - (2x + 3)(x + 2) + (2x + 3)(1 - x) \\ &= -6x^2 - 10x + 1. \end{aligned}$$

THEOREM III. The derivative of the quotient of two functions is equal to the fraction whose denominator is the square of the divisor, whilst the numerator is equal to the divisor times the derivative of the dividend minus the dividend times the derivative of the divisor.

Proof. Let y be given as the quotient of two functions u and v :

$$y = \frac{u}{v}.$$

We shall show that

$$y' = \frac{v u' - u v'}{v^2}.$$

In fact,

$$\frac{\Delta y}{\Delta x} = \frac{\frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}}{\Delta x} = \frac{v \Delta u - u \Delta v}{\Delta x v (v + \Delta v)} = \frac{v \frac{\Delta u}{\Delta x} - u \frac{\Delta v}{\Delta x}}{v (v + \Delta v)},$$

and we obtain on passing to the limit as $\Delta x \rightarrow 0$:

$$y' = \frac{v \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} - u \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x}}{v(v + \lim_{\Delta x \rightarrow 0} \Delta v)} = \frac{v u' - u v'}{v^2}.$$

This is what we had to prove.

Example.

$$\begin{aligned} y &= \frac{2-3x}{2+x}. \\ y' &= \frac{(2+x)(2-3x)' - (2-3x)(2+x)'}{(2+x)^2} \\ &= \frac{-3(2+x) - (2-3x)}{(2+x)^2} = -\frac{8}{(2+x)^2}. \end{aligned}$$

Let us use rule III to find the derivative of a power function with negative integral index $y = x^{-n}$, $n > 0$. We have:

$$\begin{aligned} y' &= (x^{-n})' = \left(\frac{1}{x^n}\right)' = \frac{x^n(1)' - 1 \cdot (x^n)'}{x^{2n}} = \frac{n x^{n-1}}{x^{2n}} \\ &= -n \frac{1}{x^{n+1}} = -n x^{-n-1}. \end{aligned}$$

Thus the general form of the derivative of a power function with positive integral exponent (Sec. 43) is now proved valid for any integral exponent.

We pass to the differentiation of two commonly encountered fractions:

(1) $y = c/u$, where c is a constant and u a function of x . We have:

$$y' = \frac{u c' - c u'}{u^2} = -c \frac{u'}{u^2};$$

this result should be memorized; in particular, with $u = x$:

$$\left(\frac{c}{x}\right)' = -\frac{c}{x^2}.$$

(2) $y = u/c$, where c is a constant. At first sight there is a temptation to differentiate this by the rule for differentiating a fraction. This should not be done. The problem is far simpler: it should be

noticed that the function can be written as the product of a constant $1/c$ and the function u :

$$y = \frac{1}{c}u;$$

hence

$$y' = \left(\frac{1}{c}u\right)' = \frac{1}{c}u' = \frac{u'}{c}.$$

Of course, the rule for differentiating a fraction leads to the same result.

47. Differentiation of a function of a function. The rule for differentiating a function of a function is of great importance; it gives the derivative in terms of the derivatives of the functions of which it is composed. Our proofs imply that the derivative of the function of a function exists if the derivatives of the component functions exist.

THEOREM. The derivative of a function of a function is equal to the derivative of the function with respect to the intermediate argument multiplied by the derivative of this argument with respect to the independent variable.

Proof. Let $y = f(u)$, $u = \varphi(x)$. We show that

$$y' = f'(u) u' = f'(u) \varphi'(x).$$

We give x the increment Δx ; it produces an increment Δu in the intermediate argument $u = \varphi(x)$, which in turn produces an increment Δy of the function y .

To find the derivative y' , we have to find $\lim \Delta y / \Delta x$ as $\Delta x \rightarrow 0$. Suppose that the increment $\Delta u \neq 0$ for all sufficiently small Δx ; then the ratio $\Delta y / \Delta x$ can be written as

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x},$$

and by the familiar rule for passage to the limit in a product (Sec. 32), we get

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x}$$

($\Delta u \rightarrow 0$ as $\Delta x \rightarrow 0$, since $u = \varphi(x)$ is a continuous function).

Since

$$\lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} = f'(u) \quad \text{and} \quad \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} = \varphi'(x),$$

we arrive at the required formula. It also holds for the case we excluded above, when $\Delta u = 0$ (see Sec. 52). In this case $y' = 0$.

Thus, to differentiate a function of a function $y = f[\varphi(x)]$, we have to take the derivative of the ("outer") function f , regarding its argument $\varphi(x) = u$ simply as a variable with respect to which the differentiation is carried out, and multiply by the derivative of the ("inner") function $\varphi(x)$ with respect to the independent variable.

Examples. (1) $y = (2x^2 - 1)^3$. We have by our rule:

$$\begin{aligned} y' &= (u^3)' \cdot u' = (u^3)' (2x^2 - 1)' = 3u^2 \cdot 4x = 3(2x^2 - 1)^2 \cdot 4x \\ &= 48x^5 - 48x^3 + 12x. \end{aligned}$$

It can easily be verified that this result is correct. On expanding the cube of the difference, we get: $y = 8x^6 - 12x^4 + 6x^2 - 1$; differentiation of this polynomial gives the same answer.

(2) $y = \left(x + \frac{1}{x}\right)^{100}$. Here we have the function $u = x + \frac{1}{x}$ to the power of 100.

We find for the derivative:

$$\begin{aligned} y' &= 100u^{99} \cdot u' = 100\left(x + \frac{1}{x}\right)^{99} \cdot \left(x + \frac{1}{x}\right)' \\ &= 100\left(x + \frac{1}{x}\right)^{99} \cdot \left(1 - \frac{1}{x^2}\right). \end{aligned}$$

The reader will appreciate the full advantage of differentiating this function with the aid of the rule for differentiating a function of a function if he compares its possible differentiation as a polynomial obtained by applying the binomial expansion to a power of 100.

Now let y be conveniently written as a function of x by means of a chain of three, instead of two, functions:

$$y = f(u), \quad u = \varphi(v), \quad v = \psi(x).$$

We have by the rule proved:

$$y' = f'(u) \cdot u'$$

where u' is the derivative of u , regarded as a function of the independent variable x . By the same rule:

$$u' = \varphi'(v) \cdot v' = \varphi'(v) \cdot \psi'(x).$$

On substituting this expression in the previous equation, we get

$$y' = f'(u) \cdot \varphi'(v) \cdot \psi'(x).$$

The formula may be deduced in the same way for any number of intermediate arguments. The derivative y' is always obtained as the derivative with respect to the first intermediate argument multiplied by the derivative of the first with respect to the second, multiplied by the derivative of the second with respect to the third and so on, and finally, multiplied by the derivative of the last intermediate argument (counting from y to x) with respect to x .

We can say briefly that:

The derivative of a composite function is equal to the product of the derivatives of the component functions.

Example.

$$y = \left[\frac{1 - (1 - x^2)^2}{1 + (1 - x^2)^2} \right]^3.$$

The function is obtained by raising to the third power a linear rational function of the square $(1 - x^2)^2$. We distinguish the intermediate arguments as:

$$y = u^3, \quad u = \frac{1 - v}{1 + v}, \quad v = w^2, \\ w = 1 - x^2.$$

We find by using the rule:

$$\begin{aligned} y' &= (u^3)' u' v' w' = (u^3)' \left(\frac{1 - v}{1 + v} \right)' (w^2)' (1 - x^2)' \\ &= 3u^2 \frac{-2}{(1 + v)^2} 2w(-2x) = 24u^2 \frac{xw}{(1 + v)^2}. \end{aligned}$$

Substitution for w , v and u by their expressions in terms of x gives

$$y' = 24 \left[\frac{1 - (1 - x^2)^2}{1 + (1 - x^2)^2} \right]^2 \frac{x(1 - x^2)}{[1 + (1 - x^2)^2]^2}.$$

As experience is gained in differentiation it will become unnecessary to introduce special notation for the intermediate arguments. Differentiation of functions is usually carried out by distinguishing only mentally the elementary branches leading from y to x .

48. Derivatives of the basic elementary functions. We shall now find the derivatives of all the basic elementary functions; we start with the trigonometric functions.

I. THE DERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS. Let

$$y = \sin x.$$

Let x receive the increment Δx ; we shall write for brevity $\Delta x = h$.
Now:

$$\begin{aligned}\Delta y &= \sin(x+h) - \sin x = 2 \sin \frac{x+h-x}{2} \cos \frac{x+h+x}{2} \\ &= 2 \sin \frac{h}{2} \cos \left(x + \frac{h}{2}\right).\end{aligned}$$

Consequently,

$$\frac{\Delta y}{h} = \frac{\sin \frac{h}{2}}{\frac{h}{2}} \cos \left(x + \frac{h}{2}\right)$$

and

$$y' = \lim_{h \rightarrow 0} \frac{\Delta y}{h} = \lim_{h \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \lim_{h \rightarrow 0} \cos \left(x + \frac{h}{2}\right).$$

The first factor is equal to unity by what was proved in Sec. 40, whilst

$$\lim_{h \rightarrow 0} \cos \left(x + \frac{h}{2}\right) = \cos(x+0) = \cos x,$$

since $\cos x$ is a continuous function.

Altogether,

$$y' = (\sin x)' = \cos x.$$

The derivatives of the remaining trigonometric functions are easily found by the rules for differentiation.

Let us take $y = \cos x$. Since $\cos x = \sin(x + \frac{1}{2}\pi)$, we find on putting $x + \frac{1}{2}\pi = u$:

$$y' = (\sin u)' u' = \cos u \cdot 1 = \cos \left(x + \frac{\pi}{2}\right) = -\sin x,$$

i.e.

$$y' = (\cos x)' = -\sin x.$$

Further:

$$\begin{aligned}
 (\tan x)' &= \left(\frac{\sin x}{\cos x} \right)' = \frac{(\sin x)' \cos x - (\cos x)' \sin x}{\cos^2 x} \\
 &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}, \\
 (\operatorname{cosec} x)' &= \left(\frac{1}{\sin x} \right)' = -\frac{(\sin x)'}{\sin^2 x} = -\frac{\cos x}{\sin^2 x}, \\
 (\sec x)' &= \left(\frac{1}{\cos x} \right)' = -\frac{(\cos x)'}{\cos^2 x} = \frac{\sin x}{\cos^2 x}, \\
 (\cot x)' &= \left(\frac{\cos x}{\sin x} \right)' = \frac{(\cos x)' \sin x - (\sin x)' \cos x}{\sin^2 x} \\
 &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}.
 \end{aligned}$$

II. DERIVATIVES OF THE INVERSE TRIGONOMETRIC FUNCTIONS.

Let

$$y = \arcsin x.$$

On giving x the increment Δx , we have for the "incremented" value of the function:

$$y + \Delta y = \arcsin(x + \Delta x).$$

These equations give us

$$x = \sin y, \quad x + \Delta x = \sin(y + \Delta y).$$

Hence

$$\Delta x = \sin(y + \Delta y) - \sin y.$$

We can therefore write:

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\sin(y + \Delta y) - \sin y} = \frac{1}{\frac{\sin(y + \Delta y) - \sin y}{\Delta y}}.$$

As $\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$ also, so that

$$y' = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{1}{\lim_{\Delta y \rightarrow 0} \frac{\sin(y + \Delta y) - \sin y}{\Delta y}}.$$

But the denominator is precisely the derivative of the sine; hence

$$y' = \frac{1}{(\sin y)'} = \frac{1}{\cos y}.$$

Since

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2},$$

we get finally:

$$y' = (\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}.$$

By virtue of the identity

$$\arcsin x + \arccos x = \frac{\pi}{2}$$

we have

$$(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}.$$

We take the positive sign in front of the roots because we have agreed always to consider the principal branches $\arcsin x$ and $\arccos x$. (We recommend the reader to explain how it follows from this agreement that we must take the $+$ in front of the root.)

By arguing as above for $y = \arcsin x$, we can write for $y = \arctan x$:

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\tan(y + \Delta y) - \tan y} = \frac{1}{\frac{\tan(y + \Delta y) - \tan y}{\Delta y}}$$

i.e.

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{1}{\lim_{\Delta y \rightarrow 0} \frac{\tan(y + \Delta y) - \tan y}{\Delta y}}, \quad y' = \frac{1}{(\tan y)'} = \cos^2 y.$$

But $\cos^2 y = 1/(1 + \tan^2 y) = 1/(1 + x^2)$, so that finally:

$$y' = (\arctan x)' = \frac{1}{1 + x^2}.$$

The identity

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$$

leads us to

$$(\operatorname{arccot} x)' = -\frac{1}{1 + x^2}.$$

Examples:

$$(1) [\sin(ax + b)]' = \cos(ax + b)(ax + b)' = a \cos(ax + b).$$

$$(2) (\tan^2 x)' = 2 \tan x (\tan x)' = \frac{2 \tan x}{\cos^2 x}.$$

$$(3) (\cos x^2)' = -\sin x^2 (x^2)' = -2x \sin(x^2).$$

$$\begin{aligned}
 (4) \left(\arctan \frac{1+x}{1-x} \right)' &= \frac{1}{1 + \left(\frac{1+x}{1-x} \right)^2} \cdot \left(\frac{1+x}{1-x} \right)' \\
 &= \frac{(1-x)^2}{(1-x)^2 + (1+x)^2} \cdot \frac{2}{(1-x)^2} = \frac{1}{1+x^2}.
 \end{aligned}$$

The derivative here has the same expression as for the derivative of $\arctan x$. This is hardly surprising, since

$$\arctan \frac{1+x}{1-x} = \arctan x + \arctan 1 \quad (|x| < 1).$$

III. DERIVATIVE OF THE LOGARITHMIC FUNCTION. Let $y = \ln x$. Now:

$$\frac{\Delta y}{h} = \frac{\ln(x+h) - \ln x}{h} = \frac{\ln\left(1 + \frac{1}{x}h\right)}{h},$$

and since (Sec. 41)

$$\lim_{\alpha \rightarrow 0} \frac{\ln(1 + \alpha k)}{\alpha} = k \lim_{k\alpha \rightarrow 0} \frac{\ln(1 + k\alpha)}{k\alpha} = k,$$

we have

$$y' = \lim_{h \rightarrow 0} \frac{\Delta y}{h} = \lim_{h \rightarrow 0} \frac{\ln\left(1 + \frac{1}{x}h\right)}{h} = \frac{1}{x}.$$

Thus

$$(\ln x)' = \frac{1}{x}.$$

If the logarithmic function is taken with a base different from e , the form of the derivative is not quite so simple. Let $y = \log_a x$. Then

$$y = \log_a x = \log_a e \ln x$$

i.e.

$$(\log_a x)' = (\log_a e \cdot \ln x)' = \log_a e (\ln x)' = \frac{\log_a e}{x} = \frac{1}{x \ln a}.$$

In particular,

$$(\log x)' = \frac{M}{x} \approx \frac{0.43429}{x} \quad (M = \log e).$$

IV. DERIVATIVE OF THE EXPONENTIAL FUNCTION. Let $y = a^x$. We have

$$\frac{\Delta a^x}{h} = \frac{a^{x+h} - a^x}{h} = a^x \frac{a^h - 1}{h},$$

and since (Sec. 41)

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = \ln a,$$

we get

$$y' = (a^x)' = a^x \ln a.$$

We obtain with $a = e$:

$$(e^x)' = e^x \ln e = e^x.$$

The function $y = ce^x$ (c constant) is remarkable inasmuch as it is the only function which never changes with differentiation.

V. DERIVATIVE OF THE POWER FUNCTION. We shall now prove that

$$(x^n)' = nx^{n-1}$$

for any n . It has so far been proved only for integral n .

This is done by writing $y = x^n$ ($x > 0$) as

$$y = x^n = e^{n \ln x}.$$

Differentiation of this function of a function gives

$$y' = (e^{n \ln x})' = e^{n \ln x} (n \ln x)' = e^{n \ln x} \frac{n}{x} = x^n \frac{n}{x} = nx^{n-1}.$$

Now let $x < 0$. We put $x = -z$, i.e. $z = -x$, $z > 0$. We now have, by the rule for differentiating a function of a function:

$$\begin{aligned} y' &= ((-1)^n z^n)' = (-1)^n n z^{n-1} z' = (-1)^n n z^{n-1} (-1) \\ &= (-1)^{n+1} n z^{n-1}; \end{aligned}$$

since $(-1)^{n+1} = (-1)^{n-1}$, we have

$$y' = n(-1)^{n-1} (z)^{n-1} = n(-z)^{n-1} = nx^{n-1}.$$

This is what we wanted to prove.

VI. BASIC TABLE OF DERIVATIVES.

$$\begin{array}{ll}
(x^n)' = n x^{n-1}, & (\tan x)' = \frac{1}{\cos^2 x}, \\
(a^x)' = a^x \ln a, & (\cot x)' = -\frac{1}{\sin^2 x}, \\
(e^x)' = e^x, & (\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \\
(\log_a x)' = \log_a e \cdot \frac{1}{x} = \frac{1}{x \ln a}, & (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}, \\
(\ln x)' = \frac{1}{x}, & (\arctan x)' = \frac{1}{1+x^2}, \\
(\sin x)' = \cos x, & (\operatorname{arccot} x)' = -\frac{1}{1+x^2}, \\
(\cos x)' = -\sin x, &
\end{array}$$

It is essential that this table be known by heart.

Examples.

$$(1) \ y = x \ln x, \quad y' = (x)' \ln x + x(\ln x)' = \ln x + 1.$$

$$(2) \ y = (\ln x)^3, \quad y' = 3(\ln x)^2 (\ln x)' = 3 \frac{(\ln x)^2}{x}.$$

$$(3) \ y = 2^{x^2-x}, \quad y' = 2^{x^2-x} \ln 2 \cdot (x^2-x)' = (2x-1) 2^{x^2-x} \ln 2.$$

$$\begin{aligned}
(4) \ y = e^{\sin(2x+1)}, \quad y' &= e^{\sin(2x+1)} [\sin(2x+1)]' \\
&= e^{\sin(2x+1)} \cos(2x+1) (2x+1)' \\
&= 2 \cos(2x+1) e^{\sin(2x+1)}.
\end{aligned}$$

$$\begin{aligned}
(5) \ y = \ln(x + \sqrt{1+x^2}), \quad y' &= \frac{1}{x + \sqrt{1+x^2}} (x + \sqrt{1+x^2})' \\
&= \frac{1}{x + \sqrt{1+x^2}} \left(1 + \frac{2x}{2\sqrt{1+x^2}} \right) = \frac{1}{\sqrt{1+x^2}}.
\end{aligned}$$

(6) Let us find the derivatives of the hyperbolic functions (see Sec. 22):

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

We have:

$$(\sinh x)' = \frac{(e^x)' - (e^{-x})'}{2} = \frac{e^x + e^{-x}}{2} = \cosh x,$$

$$(\cosh x)' = \frac{(e^x)' + (e^{-x})'}{2} = \frac{e^x - e^{-x}}{2} = \sinh x,$$

$$\begin{aligned} (\tanh x)' &= \left(\frac{\sinh x}{\cosh x} \right)' = \frac{(\sinh x)' \cosh x - (\cosh x)' \sinh x}{\cosh^2 x} \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}. \end{aligned}$$

A similarity can again be noticed here between the hyperbolic and trigonometric functions.

49. Logarithmic differentiation. Differentiation of inverse and implicit functions.

I. By using the rules for differentiation and the table of derivatives of the basic elementary functions, we can now find automatically the derivatives of any elementary functions, except for one type, the simplest representative of which is the function $y = x^x$. Such functions are described as power-exponential and include, in general, any function written as a power whose base and index both depend on the independent variable.

In order to find by the general rules the derivative of the power-exponential function $y = x^x$, we take logarithms of both sides of this equation:

$$\ln y = x \ln x, \quad x > 0.$$

Since this is an identity, the derivative of the left-hand side must be equal to the derivative of the right. We obtain by differentiating with respect to x (and not forgetting that the left-hand side is a function of a function):

$$\frac{1}{y} y' = \ln x + 1.$$

Hence

$$y' = y(\ln x + 1) = x^x(\ln x + 1).$$

The operation consisting in first taking the logarithm of the function $f(x)$ (to base e) then differentiating is called logarithmic differentiation and its result

$$(\ln f(x))' = \frac{f'(x)}{f(x)}$$

is called the logarithmic derivative of $f(x)$.

Logarithmic differentiation can be used for finding other derivatives besides those of power-exponential functions; it can make for shorter working in these other cases. For instance, we can use logarithmic differentiation to find the derivative of

$$y = \sqrt{x^2 + 4} \sin^2 x \cdot 2^x$$

more rapidly.

We have:

$$\ln y = \frac{1}{2} \ln(x^2 + 4) + \ln \sin^2 x + x \ln 2,$$

$$y' = y \left(\frac{x}{x^2 + 4} + 2 \cot x + \ln 2 \right).$$

II. The method which we used for finding the derivatives of the inverse trigonometric functions may be generalized to the case of any two mutually inverse functions.

THEOREM. *If $y = \varphi(x)$ is a continuous function inverse to a function $y = f(x)$ which is continuous and has a derivative, the derivative $\varphi'(x)$ exists and its value is equal to the reciprocal of the value of $f'(y)$ at $y = \varphi(x)$:*

$$\varphi'(x) = \frac{1}{f'(y)}.$$

It is assumed here that, at the point y ,

$$f'(y) \neq 0.$$

Proof. The equations

$$y = \varphi(x) \quad \text{and} \quad y + \Delta y = \varphi(x + \Delta x)$$

yield

$$x = f(y) \quad \text{and} \quad x + \Delta x = f(y + \Delta y).$$

Hence $\Delta x = f(y + \Delta y) - f(y)$. Thus

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{f(y + \Delta y) - f(y)} = \frac{1}{\frac{f(y + \Delta y) - f(y)}{\Delta y}}.$$

Since $\Delta y \rightarrow 0$ as $\Delta x \rightarrow 0$, we have

$$y' = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{1}{\lim_{\Delta y \rightarrow 0} \frac{f(y + \Delta y) - f(y)}{\Delta y}},$$

i.e.

$$y' = \frac{1}{f'(y)}.$$

This is what we wanted to prove.

The expression for the derivative $\varphi'(x)$ is obtained from $1/f'(y)$ by replacing y by its expression in terms of x , i.e. $y = \varphi(x)$. It must be carefully borne in mind that $f(y)$ must be first differentiated with respect to y , then y later replaced by $\varphi(x)$.

By using the relationship between the derivatives of two mutually inverse functions, it is easy to find one from the other. We recommend the reader to find the derivatives of $\ln x$, e^x , $\arcsin x$, $\sin x$, $\arctan x$, $\tan x$ from the derivatives of e^x , $\ln x$, $\sin x$, $\arcsin x$, $\tan x$, $\arctan x$.

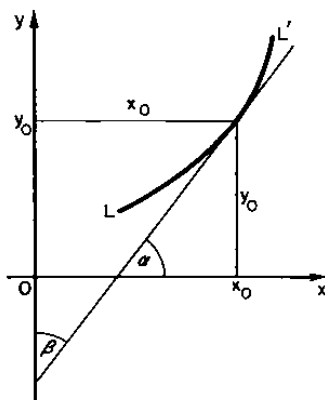


FIG. 57

The connection between the derivatives of mutually inverse functions may be illustrated very clearly geometrically. Let the curve LL' be the graph of $y = \varphi(x)$ (Fig. 57). Then

$$\varphi'(x_0) = \tan \alpha.$$

The equation of LL' can be written as: $x = f(y)$. The derivative of this function with respect to variable y at $y = y_0 = \varphi(x_0)$ is evidently equal to the slope of the same tangent, but with respect to Oy , i.e.

$$f'(y_0) = \tan \beta.$$

But $\alpha + \beta = \frac{1}{2}\pi$, i.e. $\tan \beta = \cot \alpha = 1/\tan \alpha$, whence

$$\varphi'(x_0) = \frac{1}{f'(y_0)}.$$

This formula really expresses the obvious fact that the sum of the angles formed by the tangent to a curve with the co-ordinate axes is equal to $\frac{1}{2}\pi$.

III. If y is an implicit function of x , i.e. is given by an equation which is not solved with respect to y , its derivative must be found by differentiating both sides of the equation with respect to x and taking into account that y is a function of x (defined by this equation).

Some examples will show the reader the practical meaning of this rule.

Let us find the derivative of the function y given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We obtain on differentiating with respect to x and recalling that y is a function of x :

$$\frac{2x}{a^2} + \frac{2y}{b^2} y' = 0,$$

whence

$$y' = -\frac{b^2}{a^2} \frac{x}{y}.$$

The explicit expression for y is easily found in this example, namely,

$$y = \frac{b}{a} \sqrt{a^2 - x^2}.$$

On substituting this in the expression for the derivative, we get

$$y' = -\frac{b}{a} \frac{x}{\sqrt{a^2 - x^2}}$$

— a result which is easily seen to coincide with that obtained by the direct method.

Let us further consider the equation (cf. Sec. 13):

$$xy - e^x + e^y = 0.$$

Differentiation gives

$$y + xy' - e^x + e^y y' = 0$$

whence

$$y' = \frac{e^x - y}{e^y + x}.$$

Here, we can no longer obtain an explicit expression for y' . However, since y satisfies the original equation, the derivative can be given in a somewhat different form:

$$y' = \frac{e^x - y}{e^x - xy + x}.$$

Differentiation of $y = \varphi(x)$, the inverse function to $y = f(x)$, is in the nature of things differentiation of y as an implicit function of x , given by the equation $x - f(y) = 0$.

50. Graphical differentiation. We mainly have recourse to finding a derivative graphically—necessarily approximately—when the analytic expression for the function is unknown and the function is specified graphically (say by a recording device).

Finding a derivative graphically is called graphical differentiation. Graphical differentiation is normally used in a system of Cartesian co-ordinates.

Let AB (Fig. 58) be the graph of a function $y = f(x)$. We mark off on the axis of abscissae, to the left of the origin, a segment OP equal to the scale unit.

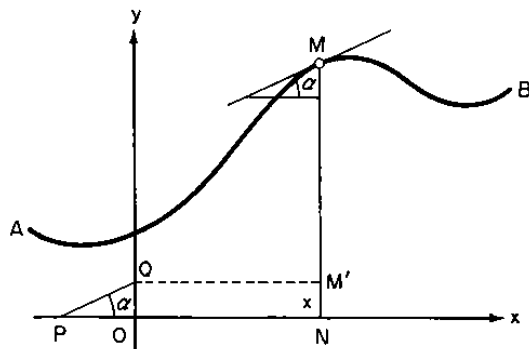


FIG. 58

We draw “by eye” the tangent to the curve AB at the point M corresponding to a given abscissa x , and draw from P (sometimes called the pole of the graph) a straight line parallel to the tangent, as far as its intersection at Q with the axis of ordinates. The segment OQ will give the required numerical derivative $f'(x)$. For,

$$OQ = OP \tan \alpha = 1 \cdot \tan \alpha = f'(x).$$

The operation of drawing the tangent “by eye” is extremely inaccurate. It can be made more exact if use is made of a special device, the “mirror derivator”, which serves to draw the normal to the curve.

A simple “mirror derivator” can be made with the aid of an ordinary ruler, with a small mirror fixed to one end. On placing an edge of the ruler against the graph at the point M (with the plane of the ruler perpendicular to the plane of the figure), it has to be rotated until the image of the curve in the mirror continues the curve itself without a break. The position of the ruler now gives the direction of the normal at M , and the perpendicular to this is the tangent.

We draw from Q a straight line parallel to Ox as far as its intersection with the ordinate MN , or its continuation, at the point M' . The ordinate of this point in fact gives the value of the numerical derivative $f'(x)$ at the given value of the independent variable x . Evidently, the points M' trace out the graph of the derivative $f'(x)$ for a given graph of the function $y = f(x)$.

Let us subdivide the piece AB of the curve $y = f(x)$ (Fig. 59), corresponding to the interval of variation of the independent variable which interests us, into several parts by the straight lines $x = x_1, x = x_2, \dots$. We now find graphically the values of the derivative at the points bisecting each sub-interval. The mid-points of the sub-intervals are suitable because, in normal cases, the tan-

gents can be taken with considerable accuracy as simply parallel to the chords joining the ends of each piece of arc of the graph.

We find further, by the method just described, the points $M'_1, M'_2, M'_3, \dots$ of the graph of the derivative. On joining these points by a continuous curve, we in fact obtain the approximate graph of the derivative $f'(x)$. It will generally speaking reproduce the derivative the more accurately, the greater the number of points M' that we find, i.e. the greater the number of parts into which we

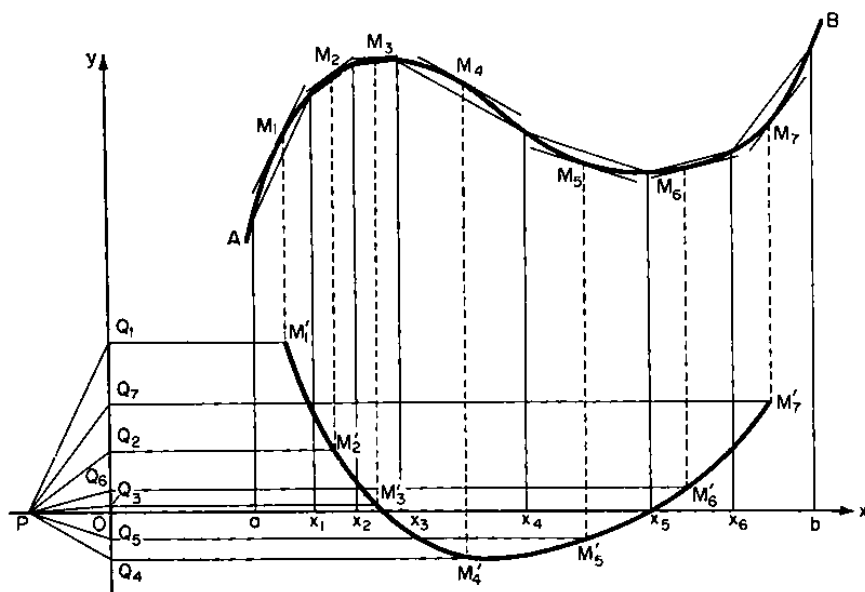


FIG. 59

subdivide the entire interval. These parts need not necessarily be equal to each other; their dimensions should be chosen so that the corresponding parts of the curve deviate as little as possible from segments of straight lines. An interval where the curve is steep and often bends should be subdivided into a greater number of parts so that each part is sufficiently small.

We have assumed throughout that the scales on the axis of abscissae and axis of ordinates are the same. In this case $f'(x) = \tan \alpha = \tan \angle TMR$ (see Fig. 53).

3. Differentials. Differentiability of a Function

51. Differentials and their geometrical interpretation. Another fundamental concept of mathematical analysis, that of the differential of a function, is very closely connected with the concept of derivative.

Let $y = f(x)$ be a continuous function in a neighbourhood of the point x . The increment Δy of the function corresponding to an

arbitrarily varying increment Δx of the independent variable is proportional to Δx in one and only one case, namely, when $f(x)$ is linear: $f(x) = ax + b$; then $\Delta y = a\Delta x$ (Sec. 17). We have already observed that the simplicity of this connection between Δy and Δx is extremely convenient. And as a matter of fact, it is in general actually possible to choose, for a given x , a constant coefficient a such that the expression $a\Delta x$, whilst not being accurately equal to Δy , nevertheless differs from Δy only by an infinitesimal of higher order than Δx (assuming that Δx , and hence also Δy , are infinitesimals):

$$\Delta y = a\Delta x + \alpha \quad (*)$$

where

$$\lim_{\Delta x \rightarrow 0} \frac{\alpha}{\Delta x} = 0.$$

We shall give a little later a very simple characteristic of the class of functions admitting the existence of such a coefficient a .

The product $a\Delta x$ is called the differential of the function $y = f(x)$ at the point x and is denoted by dy or $df(x)$:*

$$dy = a\Delta x.$$

The same remark applies to this notation as to the notation Δy , viz. that dy is not the product of some magnitude d with y , but is a unified symbol that cannot be split up.

Let us compare Δy and dy . If $a = 0$, Δy is an infinitesimal of higher order than Δx , whilst the differential dy , equal to zero, is not comparable with any other infinitesimal, Δy included. When $a \neq 0$, dy and Δy are equivalent infinitesimals or, what amounts to the same thing, dy is the principal part of Δy . For we have:

$$\frac{\Delta y}{dy} = \frac{dy + \alpha}{dy} = 1 + \frac{\alpha}{dy}$$

but $\alpha/dy = \alpha/a\Delta x = (1/a)(\alpha/\Delta x)$ tends to zero by hypothesis as $\Delta x \rightarrow 0$, i.e. $\Delta y/dy \rightarrow 1$ (see Sec. 39). We can therefore say that dy is either the principal part of Δy , proportional to Δx (assuming that $\Delta x \rightarrow 0$), or else is zero.

However, we can define the differential without providing for what is in fact a particular case, namely $a = 0$ (though this fact must be borne in mind).

* The term "differential" comes from the Latin word "differentia" meaning "difference". This underlines that the differential of a function represents, as it were, its increment, i.e. the difference between two of its values.

Definition. The *differential of a function* is the principal part of the increment of the function, proportional to the increment of the independent variable.

The expression for the differential follows from the next theorem.

THEOREM. The differential of a function, for a given value of the independent variable, is equal to the value of the derivative at this value, multiplied by the differential of the independent variable:

$$dy = f'(x)dx \quad (**)$$

Proof. Suppose that the function $y = f(x)$ has a differential at the point x , i.e. a principal part, proportional to Δx , can be distinguished in the increment of the function $\Delta y = f(x + \Delta x) - f(x)$, i.e. we can write equation (*).

Let us find the coefficient of proportionality a . This is done by dividing both sides by Δx and passing to the limit as $\Delta x \rightarrow 0$:

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = a + \lim_{\Delta x \rightarrow 0} \frac{\alpha}{\Delta x}.$$

Since the right-hand side is equal to a , the second term being zero by hypothesis, this means that the derivative of $f(x)$ exists at the point x , and $f'(x) = a$, i.e. *the coefficient of proportionality in the expression for the differential is the derivative*. Such is the direct connection holding between the concepts of derivative and differential. Thus

$$dy = f'(x)\Delta x.$$

The differential of the independent variable dx is defined as the increment Δx itself:

$$dx = \Delta x.$$

This agrees with the fact that the differential of the function $y = x$ is equal to $dy = dx = (x)' \Delta x$, i.e. $dx = 1 \cdot \Delta x$. We have thus arrived at equation (**).

The proof has shown that, if the function has a differential, it also has a derivative. We shall now prove that the converse holds:

If a function has a derivative for a given x , it also has a differential.

In fact, let $\lim_{\Delta x \rightarrow 0} \Delta y / \Delta x = f'(x)$. This implies that $\Delta y / \Delta x = f'(x) + \alpha_1$, where $\alpha_1 \rightarrow 0$ as $\Delta x \rightarrow 0$. Hence

$$\Delta y = f'(x)\Delta x + \alpha_1 \Delta x.$$

Since the second term on the right-hand side is an infinitesimal of higher order than Δx , i.e. $\alpha_1 \Delta x / \Delta x = \alpha_1 \rightarrow 0$ as $\Delta x \rightarrow 0$, the first

term is the principal part of the increment Δy (provided $f'(x) \neq 0$); it evidently always differs from Δy by an amount which is an infinitesimal of higher order than Δx ; whilst its proportionality to differential dx is obvious. Consequently, the first term $f'(x)\Delta x$ is, by definition, the differential of the function.

If we know the derivative, the differential is easily found, and conversely. The operations of finding the derivative and differential of a given function therefore have a common name, "differentiation".

In the general case (with arbitrary x), the differential $dy = f'(x)dx$ is a function of two independent variables: x and dx .

We have for the derivative:

$$f'(x) = \frac{dy}{dx}.$$

The expression of the derivative as a ratio of differentials is extremely useful in analysis. This will very soon be evident.

It follows from the above that

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

This should not be read as Δy tends to dy , and Δx to dx , as $\Delta x \rightarrow 0$. It is true here that the ratio of the increments $\Delta y/\Delta x$ tends to the ratio of the differentials dy/dx , but this only has a meaning by virtue of our definition of the differential dy .

If the differential dy of a function exists at a point x , we obtain, by taking it instead of the true increment Δy , an approximate expression with "unlimited accuracy"; in fact, $dy = f'(x)dx$ is approximately equal to Δy in a sufficiently small neighbourhood of the point x , with as small a relative error as desired. At the same time, we preserve the simplicity of the expression for the increment of the function (i.e. the proportionality to Δx) which holds accurately only for a linear function.

We pass to the geometrical interpretation of the differential of the function $y = f(x)$ (Fig. 60).

Since $f'(x) = \tan \alpha$ (Fig. 53), the differential $dy = f'(x)dx$ measures the segment RT , i.e.

The differential $df(x)$ of a function $f(x)$ at a point x is represented by the increment of the ordinate of the tangent to the curve $y = f(x)$ at the corresponding point $(x, f(x))$.

The increment $\Delta f(x)$ of the function is represented by the increment of the ordinate of the curve (segment RM' in Fig. 60).

Hence the difference between the differential and the increment is represented by the segment $M'T$, lying between the curve and the tangent to it; *this segment is an infinitesimal of higher order than the segment MR as $dx \rightarrow 0$.*

The differential of a function at a given point may be either larger than the increment (as in Fig. 60), or smaller than it.

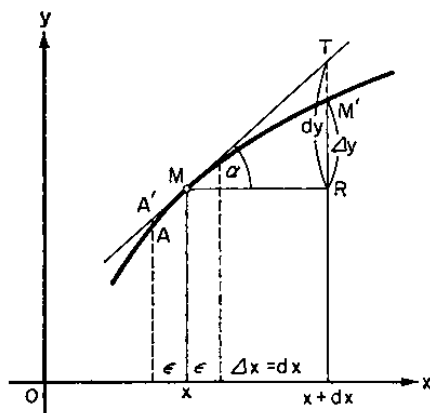


FIG. 60

52. Properties of the differential. In view of the connection between the derivative and differential, the table of derivatives of the basic elementary functions leads at once to a similar table for the differentials, and the rules for finding the derivative to corresponding rules for finding the differential.

I. We have the table of differentials (see Sec. 48):

$dx^n = n x^{n-1} dx,$	$d \tan x = \frac{1}{\cos^2 x} dx,$
$da^x = a^x \ln a dx,$	$d \cot x = -\frac{1}{\sin^2 x} dx,$
$de^x = e^x dx,$	$d \arcsin x = \frac{1}{\sqrt{1-x^2}} dx,$
$d \log_a x = \log_a e \frac{1}{x} dx,$	$d \arccos x = -\frac{1}{\sqrt{1-x^2}} dx,$
$d \ln x = \frac{1}{x} dx,$	$d \arctan x = \frac{1}{1+x^2} dx,$
$d \sin x = \cos x dx,$	$d \operatorname{arccot} x = -\frac{1}{1+x^2} dx.$
$d \cos x = -\sin x dx.$	

II. Corresponding to what was proved in Sec. 46:

(a) if $y = u + v + \dots + w$, then $dy = du + dv + \dots + dw$;

(b) if $y = uv$, then $dy = u dv + v du$, and in particular, $d cu = c du$, where c is constant;

(c) if $y = u/v$, then $dy = (v du - u dv)/v^2$.

III. Before considering the property of the differential that follows from the rule for differentiating a function of a function, we can now give a deduction of this rule which, unlike the previous (see Sec. 47), is completely general.

Let $y = f(u)$ and $u = \varphi(x)$ be continuous functions of their arguments, having derivatives with respect to these arguments: $f'(u)$ and $\varphi'(x)$. We give x the increment Δx ; then u receives the increment Δu , which in turn produces the increment Δy of y .

We can write (Sec. 51):

$$\Delta y = f'(u) \Delta u + \alpha_1 \Delta u,$$

where $\alpha_1 \rightarrow 0$ as $\Delta u \rightarrow 0$ (we can assume that $\alpha_1 = 0$ if $\Delta u = 0$).

On dividing all the terms of this equation by Δx , we get

$$\frac{\Delta y}{\Delta x} = f'(u) \frac{\Delta u}{\Delta x} + \alpha_1 \frac{\Delta u}{\Delta x}.$$

On passing to the limit as $\Delta x \rightarrow 0$:

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = f'(u) \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \rightarrow 0} \alpha_1 \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x}$$

and observing that, if $\Delta x \rightarrow 0$, $\Delta u \rightarrow 0$, i.e. $\alpha_1 \rightarrow 0$ also, we find:

$$y' = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = f'(u) \cdot u' = f'(u) \varphi'(x). \quad (*)$$

This is what we had to prove.

IV. THEOREM. The differential of a function $y = f(u)$ retains the same expression independently of whether its argument U is the independent variable or is a function of the independent variable.

Proof. We have from equation (*):

$$dy = f'(u) \cdot \varphi'(x) dx;$$

but $\varphi(x) dx = d\varphi(x) = du$, so that $dy = f'(u) du$.

We have obtained the same expression for the differential as would have been got if the argument u had been the independent variable. This is what we wanted to prove.

The property established by this theorem is called the invariance of the form of the differential. By virtue of this property, we can always write the differential of a function in the same form, without regard to the nature of the argument of the function.

Thus the table of differentials given above is valid not only when x is the independent variable, but also when x is a function of the independent variable.

It follows from the equation $dy = f'(u) du$ that

$$f'(u) = \frac{dy}{du}$$

i.e. it is always true that

The rate of change of a function with respect to its argument is equal to the ratio of the differential of the function to the differential of the argument.

Equation (*) can now be rewritten as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}. \quad (**)$$

The right-hand side of (**) is obtained from the left simply by multiplying and dividing by du (obviously, if $du \neq 0$). It is therefore possible to carry out the arithmetical operations on differentials as though they were ordinary numbers. This makes it often very convenient to write the derivative as the ratio of the differentials.

For instance, this form of writing may be used as follows to deduce the rule for differentiating an inverse function (cf. Sec. 49):

$$y' = \frac{dy}{dx} = \frac{\frac{1}{\frac{dx}{dy}}}{\frac{dy}{dy}} = \frac{1}{x'}.$$

It can be shown that formula (**) is immediately obtained by direct multiplication and division by du of the expression on the right-hand side of the equation $y' = dy/dx$.

But the point is that, whilst du/dx is equal to $\varphi'(x)$ by definition (as the ratio of the differential of the function to the differential of the independent variable), we cannot directly conclude that dy/du is equal to $f'(u)$. The fact that this ratio nevertheless is equal to $f'(u)$ is precisely a consequence of equation (*) that we have proved.

53. Application of the differential to approximations. The property of the differential enables us to replace the accurate expression for $\Delta y = f(x_0 + \Delta x) - f(x_0)$, usually very complicated, by the extremely simple expression $f'(x_0) dx$, the finding of which amounts to

differentiation. This latter operation, as we have seen, is easily accomplished for any elementary function.

Thus, for small Δx ,

$$\Delta y = f(x_0 + \Delta x) - f(x_0) \approx f'(x_0) \Delta x = dy. \quad (\dagger)$$

If we take $\Delta y = dy$, we replace the given function $y = f(x)$, where $x = x_0 + \Delta x$, by the linear function

$$y = f(x_0) + f'(x_0)(x - x_0),$$

defined by the fact that its value and the value of its derivative at $x = x_0$ are equal respectively to $f(x_0)$ and $f'(x_0)$.

Geometrically, this is equivalent to replacing the curve representing the graph of $y = f(x)$ by the tangent to it at the point $M(x_0, f(x_0))$. Generally speaking this substitution leads to an error which can be neglected in the problem when the neighbourhood of x_0 is sufficiently small. The defect of our formula lies in the lack of an indication of the maximum error characterizing its accuracy. We know that, as $\Delta x \rightarrow 0$, the relative error tends to zero, but we cannot estimate it for a given numerical value of Δx . This defect will shortly be eliminated (see Sec. 79).

Approximate equation($\dagger$) is used in practice primarily for solving problems of the two types considered below.

I. *Given the values of $f(x_0)$, $f'(x_0)$, Δx , find the approximate value of $f(x_0 + \Delta x)$.*

We have here, from($\dagger$),

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \Delta x.$$

The following examples are for illustration (writing Δx as h for the sake of brevity):

(1) $y = \sqrt[n]{1+x}$. We have:

$$dy = \frac{1}{n} \frac{\sqrt[n]{1+x}}{1+x} h,$$

i.e.

$$\sqrt[n]{1+x+h} \approx \sqrt[n]{1+x} + \frac{1}{n} \frac{\sqrt[n]{1+x}}{1+x} h;$$

in particular, when $x = 0$,

$$\sqrt[n]{1+h} \approx 1 + \frac{1}{n} h;$$

this approximate formula was previously obtained in Sec. 40 for $n = 2$.

(2) $y = \sin x$. We have:

$$dy = \cos x \cdot h$$

i.e.

$$\sin(x + h) \approx \sin x + \cos x \cdot h.$$

In particular, with $x = 0$ we find the familiar formula (Sec. 40): $\sin h \approx h$, following from the equivalence of $\sin h$ and h as $h \rightarrow 0$.

On putting say $x = \pi/6$ ($= 30^\circ$), $h = \pi/180$ ($= 1^\circ$), we find:

$$\sin 31^\circ \approx \sin 30^\circ + \cos 30^\circ \cdot \frac{\pi}{180} = 0.5 + \frac{\sqrt{3}}{2} \cdot 0.01745 \approx 0.5151;$$

$\sin 31^\circ \approx 0.51504$ to five correct places. The reader should explain why we have obtained a high approximate value for $\sin 31^\circ$.

(3) $y = \ln x$. We have:

$$dy = \frac{1}{x} h$$

i.e.

$$\ln(x + h) \approx \ln x + \frac{h}{x},$$

in particular, with $x = 1$ we arrive at the familiar formula (Sec. 41): $\ln(1 + h) \approx h$, following from the equivalence of $\ln(1 + h)$ and h as $h \rightarrow 0$.

Suppose we know that $\ln 781 \approx 6.66058$; let us evaluate $\ln 782$. The formula gives:

$$\ln 782 \approx 6.66058 + \frac{1}{781} \approx 6.66186.$$

We find from five figure tables: $\ln 782 = 6.66185$. It will be seen that the error we have made is insignificant, though we have erred on the high side. Again, the reader should consider why this must be so.

II. Given the values of $f(x_0)$, $f'(x_0)$; find the maximum error ε of $f(x_0)$ as an approximation to $f(x_0 + \Delta x)$ for a given maximum error δ in the value of x_0 ($|\Delta x| < \delta$).

We have here, from (†):

$$|\Delta y| \approx |f'(x_0)| |\Delta x| < |f'(x_0)| \delta = \varepsilon$$

i.e. the maximum error in the value of $f(x_0)$ is equal to the absolute value of the derivative $f'(x_0)$ multiplied by the maximum error in the argument.

It is also clear from this what the maximum error δ must be in specifying the argument x_0 for a previously assigned permissible error ε to be guaranteed in the value of $f(x_0)$:

$$\delta = \frac{\varepsilon}{|f'(x_0)|}.$$

The relative error in $f(x_0)$ is equal to

$$\frac{\varepsilon}{|f(x_0)|} = \left| \frac{f'(x_0)}{f(x_0)} \right| \delta,$$

i.e. the relative maximum error in the value of $f(x_0)$ is equal to the absolute value of the logarithmic derivative $f'(x_0)/f(x_0)$ multiplied by the maximum error in the argument.

We shall illustrate with some examples.

(1) In a right-angled triangle, we know the hypotenuse c , and the approximate value α of the acute angle to an accuracy of δ , i.e. $|\Delta\alpha| < \delta$. Let us find the relative errors in evaluating the adjacent side a opposite the angle α , and the other side b , assuming that $\Delta\alpha$ and δ are small.

We have:

$$a = c \sin\alpha, \quad b = c \cos\alpha,$$

whence

$$\left| \frac{da}{a} \right| = \cot\alpha \cdot |\Delta\alpha| < \cot\alpha \cdot \delta, \quad \left| \frac{db}{b} \right| = \tan\alpha \cdot |\Delta\alpha| < \tan\alpha \cdot \delta.$$

These estimates show that the relative error will always be less for the greater adjacent side.

(2) Tables with n figures give the values of the logarithms to an accuracy of $1/(2 \times 10^n)$. Let us find the accuracy with which a number should be taken in order to preserve the tabulated accuracy of their common logarithms by using the tables.

The relationship

$$y = \log x$$

gives us (Sec. 48):

$$dy = \frac{M}{x} \Delta x \approx \frac{0.4343}{x} \Delta x,$$

i.e. after rounding off,

$$\frac{1}{2 \cdot 10^n} \approx \frac{0.5}{x} \Delta x \quad \text{or} \quad \frac{\Delta x}{x} \approx \frac{1}{10^n},$$

whence it follows that the relative error in the number x must not exceed $1/10^n$, in other words, x must also be given to n correct figures.

54. Differentiability of a function. Smoothness of a curve.

Definition. A function $y = f(x)$ is said to be **differentiable at a point** x if it has a differential at this point.

In this case, as we saw in Sec. 51, the derivative $f'(x)$ also exists at the point x , and conversely, if the function has a derivative for the given x , it also has a differential. The existence of the derivative can therefore be taken as a condition for the differentiability of a function.

Let $f(x)$ be differentiable for a given x ; then it is necessarily continuous at this point. In fact, the ratio $\Delta y / \Delta x = \frac{f(x + \Delta x) - f(x)}{\Delta x}$ can have a limit as $\Delta x \rightarrow 0$ only when $\Delta y = f(x + \Delta x) - f(x)$ tends to zero simultaneously with Δx , this being in fact the condition for continuity of $y = f(x)$ at the point x .

Thus a function cannot have a derivative at a point of discontinuity.

The continuity of a function is a necessary condition for its differentiability.

This condition is by no means sufficient, however. A function which is continuous for every value of the independent variable may have no derivative.

The present-day view of continuity of a function has been a fairly recent achievement of science; the subtle difference between continuous and differentiable functions was first remarked by the great Russian mathematician N. I. Lobachevskii (see Sec. 4).

Let us consider some typical examples of continuous functions having no derivative.

I. The function $y = f(x)$, given in the interval $[0, 1]$ by the equation $f(x) = x$, and in the interval $[1, 2]$ by the equation $f(x) = 2 - x$ is evidently continuous throughout the interval $[0, 2]$. At the same time it is non-differentiable at $x = 1$. For

$$\frac{\Delta y}{\Delta x} = \frac{f(1 + \Delta x) - f(1)}{\Delta x} = \begin{cases} \frac{(1 + \Delta x) - 1}{\Delta x} = 1, & \text{if } \Delta x < 0, \\ \frac{[2 - (1 + \Delta x)] - 1}{\Delta x} = -1, & \text{if } \Delta x > 0. \end{cases}$$

Consequently the limit of $\Delta y / \Delta x$ as Δx tends to zero in an arbitrary manner does not exist, so that the continuous function $f(x)$ has no derivative at $x = 1$. This means geometrically that the continuous curve $y = f(x)$ has no tangent at the point $(1, 1)$. And in

fact, the graph of our function is the step-line (Fig. 61) whose vertex is at $(1, 1)$. It is clear that the tangent does not exist at this point, and the curve has no definite direction there.

It is sometimes said conventionally in such cases that the function has "two derivatives" at a point (the curve has "two tangents"): one corresponding to values of the function taken to the left of the point, the other taken to the right. Such a situation is always found when $\Delta y/\Delta x$ tends to two different values depending on whether Δx approaches zero from the left only or from the right only. The

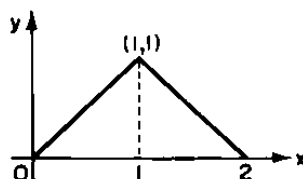


FIG. 61

points of the graph corresponding to values of x at which this occurs are called *vertices*. At such points a continuous curve changes its direction suddenly with a jump. It is clear from this that the existence of the derivative guarantees not only the continuity of the graph but also its "smoothness", i.e. the absence of vertices.

Definition. A curve is said to be *smooth* if it is continuous and has a continuously rotating tangent. Correspondingly, a function is called *smooth* if it is continuous along with its derivative. A curve (function) is said to be *piecewise smooth* if it consists of a finite number of smooth arcs (functions).

II. It is difficult, however, to visualize a continuous curve with no vertices and no tangent anywhere. Yet such curves exist. Let us take a function familiar to us:

$$f(x) = x \sin \frac{1}{x} \quad \text{for } x \neq 0 \quad \text{and} \quad f(0) = 0.$$

It is continuous everywhere, but non-differentiable at $x = 0$. In fact

$$\frac{\Delta f}{\Delta x} = \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \frac{\Delta x \sin \frac{1}{\Delta x} - 0}{\Delta x} = \sin \frac{1}{\Delta x},$$

and $\sin 1/\Delta x$ does not tend to a limit as $\Delta x \rightarrow 0$, even if Δx runs only through positive or only through negative values. On indefinitely approaching the point $(0, 0)$, the point of the graph (Fig. 62) performs an infinite set of oscillations, and even though there are no vertices in the above sense, nevertheless this continuous curve has no definite direction (tangent) at the point $(0, 0)^*$.

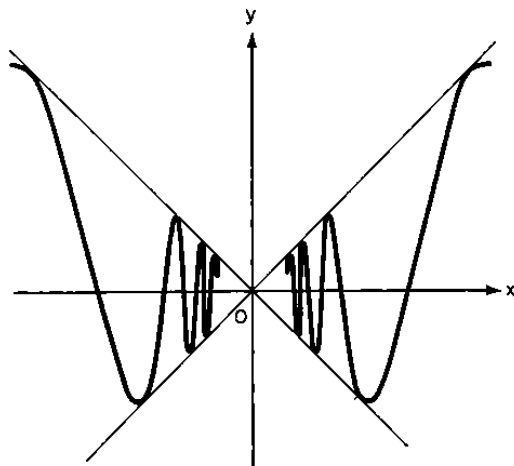


FIG. 62

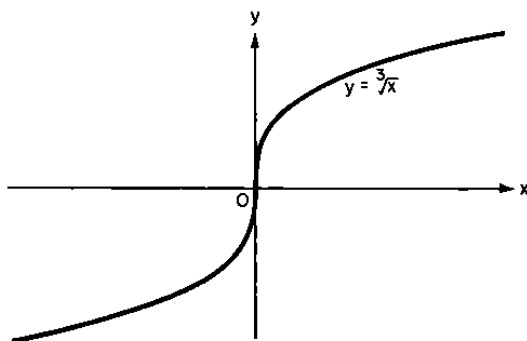


FIG. 63

III. Finally, cases are possible where $\frac{f(x + \Delta x) - f(x)}{\Delta x}$ tends to infinity as Δx tends to zero. This means geometrically that the tangent at the corresponding point is perpendicular to Ox . We

* There are examples of continuous functions, non-differentiable at every point.

therefore say for the sake of generality (in spite of the non-differentiability of the function) that in these cases the function has an infinite derivative, or derivative equal to ∞ .

(1) Let $f(x) = \sqrt[3]{x}$. We have with $x = 0$:

$$\frac{\Delta f}{\Delta x} = \frac{f(\Delta x) - f(0)}{\Delta x} = \frac{\sqrt[3]{\Delta x}}{\Delta x} = \frac{1}{\sqrt[3]{(\Delta x)^2}}$$

i.e. $\Delta f/\Delta x$ tends to $+\infty$ as Δx tends arbitrarily to zero. The curve $y = \sqrt[3]{x}$ touches Oy at the point $(0, 0)$ and has no vertices (Fig. 63). It is a smooth curve.

(2) Let $f(x) = \sqrt[3]{x^2}$. We have with $x = 0$:

$$\frac{\Delta f}{\Delta x} = \frac{\sqrt[3]{(\Delta x)^2}}{\Delta x} = \frac{1}{\sqrt[3]{\Delta x}}$$

i.e. $\Delta f/\Delta x$ tends to $-\infty$ if $\Delta x \rightarrow 0$ from the left, and to $+\infty$ if $\Delta x \rightarrow 0$ from the right. The graph of $y = \sqrt[3]{x^2}$ (a semicubical parabola) has Oy for its tangent at the point $(0, 0)$ (Fig. 64); the fact that $\Delta f/\Delta x$ tends to $-\infty$ or to $+\infty$ depending on how Δx tends to zero indicates, as in the case of a finite limit, the presence of a break in the curve, in this case a cusp, at $(0, 0)$. The semicubical parabola is said to have "two coincident tangents" at $(0, 0)$.

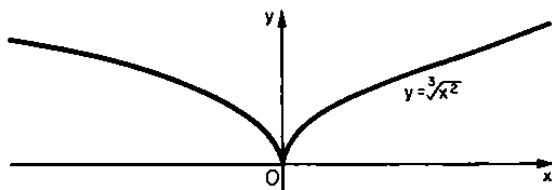


FIG. 64

Notice, however, that all the elementary functions are differentiable wherever they are defined, with the exception of individual points only, so that generally speaking, the graphs of elementary functions are continuous and smooth curves.

4. Derivative as Rate of Change (Further Examples)

55. Rate of change of a function with respect to a function. Parametric specification of functions and curves.

I. RATE OF CHANGE. Suppose we are given two functions x and y of the same variable t :

$$x = \varphi(t), \quad y = f(t).$$

The idea of the "rate of change of a function" departs from a comparison of the change of the function with the change of its argument; now, however, we shall compare the change of one function with the change of the other.

Let the value of the variable t be given an increment Δt and let us consider the change of functions φ and f in the interval $(t, t + \Delta t)$:

$$\Delta x = \varphi(t + \Delta t) - \varphi(t), \quad \Delta y = f(t + \Delta t) - f(t).$$

Definition. The rate

$$v_{av} = \Delta y / \Delta x = \frac{f(t + \Delta t) - f(t)}{\varphi(t + \Delta t) - \varphi(t)}$$

is termed the average rate of change of function $f(t)$ with respect to function $\varphi(t)$ in the interval $(t, t + \Delta t)$.

This average rate indicates the number of units of f per unit of φ in the interval $(t, t + \Delta t)$.

Definition. The limit v of the average rate v_{av} as $\Delta t \rightarrow 0$, if it exists, is called the **rate of change of function $f(t)$ with respect to function $\varphi(t)$ at the given point t^*** :

$$v = \lim_{\Delta t \rightarrow 0} v_{av} = \lim_{\Delta t \rightarrow 0} \Delta y / \Delta x.$$

Let us evaluate v . We have:

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta t \rightarrow 0} \frac{\frac{\Delta y}{\Delta t}}{\frac{\Delta x}{\Delta t}} = \lim_{\Delta t \rightarrow 0} \frac{\frac{f(t + \Delta t) - f(t)}{\Delta t}}{\frac{\varphi(t + \Delta t) - \varphi(t)}{\Delta t}} = \frac{f'(t)}{\varphi'(t)}.$$

Thus,

$$v = \frac{y'}{x'} = \frac{f'(t)}{\varphi'(t)}$$

i.e. the rate of change of one function with respect to another is equal to the ratio of the derivatives of these functions with respect to their common argument, or in other words, to the ratio of the rates of change of these functions.

* To distinguish this rate of change from the ordinary rate of change with respect to the argument, it may be called the "relative" rate of change.

The concept of the rate of change of a function (f) is a particular case of the concept just defined of relative rate of change, when the function (φ), with respect to which the rate is taken, is simply the independent variable: $\varphi(t) = t$; then

$$v = \frac{y'}{1} = f'(t).$$

II. PARAMETRIC SPECIFICATION. Specifying the equations

$$x = \varphi(t), \quad y = f(t) \quad (\dagger)$$

implies specifying a functional relationship between variables x and y . For, given the value of t (in some domain), we can find from our system values of x and y which in fact correspond to each other.

Definition. The specification of a functional relationship between two variables, whereby both variables are each defined separately as a function of the same auxiliary variable, is described as *parametric*, the auxiliary variable being the *parameter*.

The operation of finding from system($\dagger$) the direct connection between the variables x and y without the presence of variable t is called *elimination of the parameter*. As a result of eliminating the parameter, we obtain an equation between x and y which defines one as an explicit or implicit function of the other.

The straightforward way to eliminate the parameter is this: we find, say from the first equation, an expression for t in terms of x , i.e. $t = \psi(x)$, where ψ is the inverse function to φ , and on substituting this expression in the second equation:

$$y = f[\psi(x)] = F(x)$$

we get an explicit expression for y as a function of x . In actual cases various other methods of eliminating the parameter may be used (see III, examples 1, 2).

If we regard the relationship between x and y as the equation of the corresponding curve, it will be seen that the latter can be specified "parametrically", or more precisely, "by parametric equations". This means that the abscissa and ordinate of the curve are functions of the same variable—the parameter.

The parametric specification of functions and curves often has advantages over other methods of specifying them. Whereas the direct connection between x and y may be complicated, and in particular may be represented by a many-valued expression, the func-

tions defining x and y in terms of a parameter may be single-valued and simple. In addition, the parametric specification makes no proviso as to which variable in a functional relationship is to be taken as independent, and which as dependent.

It should be noticed that a function can be specified parametrically in many ways, and not just in one (see III, example 2).

As regards the parameter, our interpretation of it depends on the nature of the functional relationship and on other circumstances. Time is often taken as a parameter, in which case corresponding values of x and y are those which are taken at the same instant. In other cases the parameter is a variable arc, area, temperature etc.

We shall now show that the relative rate of change of a function $y = f(t)$ with respect to a function $x = \varphi(t)$ can be regarded as the ordinary rate of change of y as a function of the independent variable x .

In fact, system(†) of two functions x and y of a common argument t can be regarded as a parametric representation of one of them (say y) as a function of the other (of x). By what we proved in Sec. 52, the rate of change v of this function will be equal to dy/dx , and we get

$$v = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{f'(t)}{\varphi'(t)}, \quad (\dagger\dagger)$$

i.e. precisely the expression found above for the relative rate of change of f with respect to φ .

Formula (††) gives us a rule for differentiating a function specified parametrically.

It may be shown similarly that the derivative of x with respect to y is equal to $\varphi'(t)/f'(t)$.

III. EXAMPLES. We shall give some examples of the parametric specification of curves.

(1) Let us take the circle with centre at the origin and radius equal to a . We write t for the arc of the circle (in radians) from the point $(a, 0)$ to the current point (x, y) .

Obviously,

$$x = a \cos t, \quad y = a \sin t.$$

These are in fact the parametric equations of the circle. Elimination of t , by squaring x and y and adding, gives

$$x^2 + y^2 = a^2.$$

Here, y is two-valued as a function of x (and x as a function of y), whilst the parametric specification of the same relationship is established with the aid of single-valued functions.

When t runs through the interval $[0, 2\pi)$, the point with co-ordinates x, y runs once round the circle.

We have for the slope of the tangent to the circle:

$$\frac{dy}{dx} = \frac{a \cos t}{-a \sin t} = -\cot t.$$

The slope of the radius drawn to the same point of the circle is $\tan t$. Hence it follows that the tangent to the circle is perpendicular to the radius, so that the elementary definition of the tangent to a circle follows from our general definition of the tangent to a curve.

(2) Given an ellipse with centre at the origin and axes a and b lying along the co-ordinate axes, let us draw a circle with the same centre and radius equal to a (if $a > b$). If we take as parameter t the arc of this circle from the point $(a, 0)$ to the point with the same abscissa as the current point (x, y) of the ellipse, we obtain as parametric equations of the ellipse:

$$x = a \cos t, \quad y = b \sin t.$$

Elimination of parameter t from these equations leads to the familiar canonical equation of the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

As t varies in the interval $[0, 2\pi)$, the point (x, y) moves along the ellipse. It follows from this, by the way (regarding t as time), that, if the projections of a point on to two mutually perpendicular directions perform harmonic vibrations with the same periods but with phases differing by $\frac{1}{2}\pi$ (see Sec. 25), the point moves along an ellipse (in the case of unequal amplitudes) or along a circle (in the case of equal amplitudes).

Further parametric equations of the ellipse are:

$$x = \frac{a}{\cosh t}, \quad y = b \tanh t \quad (-\infty < t < +\infty).$$

(3) The parametric equations of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ are:

$$x = a \cosh t, \quad y = b \sinh t, \quad -\infty < t < \infty.$$

If we take the rectangular hyperbola $a = b = 1$, the parameter t can be given the following geometrical meaning: it is equal to twice the area of the curvilinear triangle bounded by the real semi-axis, the straight segment joining the centre to the given point of the hyperbola, and the hyperbola itself (see Sec. 117). The parameter t has a precisely similar meaning in the equations of a circle of radius 1: it is equal to twice the area of the sector of the circle, bounded by the radii joining the centre to the initial point $(1, 0)$ and to the given point of the circle. Because of this fact, the functions $\cosh t$ and $\sinh t$ (as also $\tanh t$, etc.) were called the hyperbolic functions; whilst the functions $\cos t$ and $\sin t$ (and the other trigonometric functions) are also called circular.

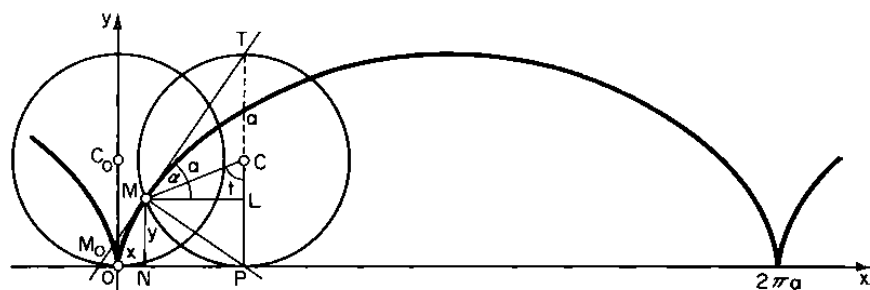


FIG. 65

(4) The cycloid is the curve described by a given point of a circle which rolls without slipping* along a straight line (Fig. 65).

The straight line along which the rolling occurs is called the "base line", whilst the rolling circle is the "generating circle".

If the base line is taken as Ox , and we take as origin the point of the line which coincides with the fixed point M_0 of the generating circle, the equation of the cycloid has the form

$$x \pm \sqrt{y(2a - y)} = a \arccos \frac{a - y}{a}, \quad 0 \leq x \leq 2\pi a,$$

where a is the radius of the circle.

The parametric equations of the cycloid are very simple:

$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$

* One curve is said to roll along another without slipping if the length of arc through which the moving curve rotates is exactly equal to the length of the piece of fixed curve which is traversed.

In fact, we take as parameter the angle t , through which the circle rotates when the fixed point passes from M_0 to the position M . This angle is equal to the angle between the radius of the rolling circle drawn to the point M , and the radius drawn to the point P at which it touches Ox . Let us find the abscissa of the point M of the cycloid:

$$x = ON = OP - NP;$$

but $OP = MP = at$ (the circle rolls without slip!), whilst $NP = ML = a \sin t$. Thus $x = a(t - \sin t)$. Similarly, the ordinate of point M is:

$$y = MN = PC - LC = a - a \cos t = a(1 - \cos t).$$

One arc of the cycloid is obtained when t varies in the interval $[0, 2\pi]$.

Let us find the slope of the tangent to the cycloid at the point M , corresponding to the value t of the angle of rotation of the generating circle:

$$\frac{dy}{dx} = \frac{a \sin t}{a(1 - \cos t)} = \cot \frac{1}{2}t;$$

i.e.,

$$\tan \alpha = \cot \frac{1}{2}t = \tan(\frac{1}{2}\pi - \frac{1}{2}t), \text{ where } \alpha = \angle TML,$$

whence

$$\alpha = \frac{\pi}{2} - \frac{t}{2} \quad \text{and} \quad \angle TMC = \frac{\pi}{2} - \frac{t}{2} - \left(\frac{\pi}{2} - t\right) = \frac{t}{2}.$$

Thus, the tangent to the cycloid passes through the "upper" point T , and the normal through the "lower" point P of the generating circle at its position corresponding to the given point of contact M .

56. Rate of change of radius vector. We have so far interpreted the independent variable and the function geometrically as corresponding to the abscissa and ordinate of a point on the plane; the derivative, measuring the rate of change of the ordinate with respect to the abscissa, was found to be equal to the slope of the tangent to the graph of the function.

We shall now interpret the independent variable and function as corresponding geometrically to the polar angle and radius vector of a point on the plane. The curve, defined in a system of polar co-ordinates (ϱ, φ) by an equation connecting the independent variable φ and the function ϱ :

$$\varrho = f(\varphi)$$

is called the graph of function f . The question arises of the geometrical meaning of the derivative $d\rho/d\varphi = f'(\varphi)$, measuring the rate of change of the radius vector with the polar angle. This is answered by the following theorem.

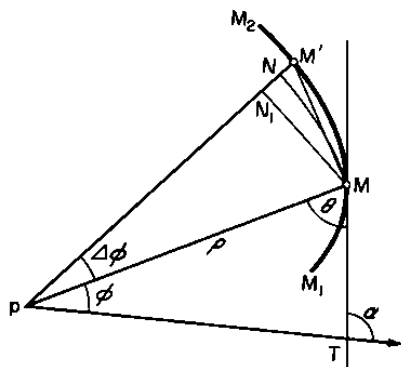


FIG. 66

THEOREM. The rate of change ρ' of the radius vector ρ of a curve $\rho = f(\varphi)$ with respect to the polar angle φ is equal to the radius vector multiplied by the cotangent of the angle θ between the radius vector and the tangent to the curve at the point in question:

$$\rho' = \rho \cot \theta.$$

Proof. Let the curve $M_1 M_2$ (Fig. 66) be the graph of the function $\rho = f(\varphi)$. Let the angle φ receive the increment $\Delta\varphi$. Then the point $M(\rho, \varphi)$ passes to the point $M'(\rho + \Delta\rho, \varphi + \Delta\varphi)$. With the pole P as centre, we draw the arc of a circle $\widetilde{MN}$ and drop from M a perpendicular MN_1 on to the radius vector PM' . Then (see Fig. 66),

$$\frac{\Delta\rho}{\Delta\varphi} = \frac{NM'}{\Delta\varphi} = \rho \frac{NM'}{\widetilde{MN}}$$

(since $\widetilde{MN} = \rho\Delta\varphi$) or

$$\frac{\Delta\rho}{\Delta\varphi} = \rho \frac{N_1 M' - N_1 N}{MN_1} \cdot \frac{MN_1}{\widetilde{MN}} = \rho \left(\frac{N_1 M'}{MN_1} - \frac{N_1 N}{MN_1} \right) \cdot \frac{MN_1}{\widetilde{MN}}, \quad (\dagger)$$

where $\frac{N_1 M'}{MN_1} = \cot \angle N_1 M' M = \cot \angle P M' M$,

$$\frac{N_1 N}{MN_1} = \frac{PN - PN_1}{MN_1} = \frac{\rho - \rho \cos \Delta\varphi}{\rho \sin \Delta\varphi} = \tan \frac{\Delta\varphi}{2},$$

$$\frac{MN_1}{\widetilde{MN}} = \frac{\rho \sin \Delta\varphi}{\rho \Delta\varphi} = \frac{\sin \Delta\varphi}{\Delta\varphi}.$$

Thus, if $\Delta\varphi \rightarrow 0$, then $N_1N/MN_1 \rightarrow 0$, $MN_1/\widetilde{MN} \rightarrow 1$, and since the secant MM' tends to the tangent MT , then $\angle PM'M \rightarrow \angle PMT = \theta$ and $N_1M'/MN_1 \rightarrow \cot\theta$.

By virtue of this, on passing to the limit in equation ($\dagger$) as $\Delta\varphi \rightarrow 0$, we get:

$$\varrho' = \varrho \cot\theta \quad (\text{or } \varrho'/\varrho = \cot\theta).$$

This is what we wanted to prove.

It is clear from this how, with the aid of the differential calculus, we can solve problems connected with tangents and normals to curves given in polar co-ordinates.

Examples. (1) Since the polar equation of a circle with centre at the origin, (the pole) has the form $p = a = \text{const}$, we have $\cot\theta = \varrho'/\varrho = 0$, i.e. $\theta = \frac{1}{2}\pi$, i.e. the tangent to a circle is perpendicular to the radius through the point of contact.

(2) Let us show that the so-called logarithmic spiral $\varrho = ae^{m\varphi}$ cuts any straight line through the pole at a constant angle (Fig. 67). In fact,

$$\varrho' = am e^{m\varphi} = m\varrho$$

i.e.

$$\cot\theta = \varrho'/\varrho = m.$$

If the spiral does not degenerate to a circle ($m \neq 0$), the angle θ always differs from a right-angle.

One logarithmic spiral can be drawn through any two points M_1 and M_2 of the plane (two parameters in the equation $\varrho = ae^{m\varphi}$); movement along this

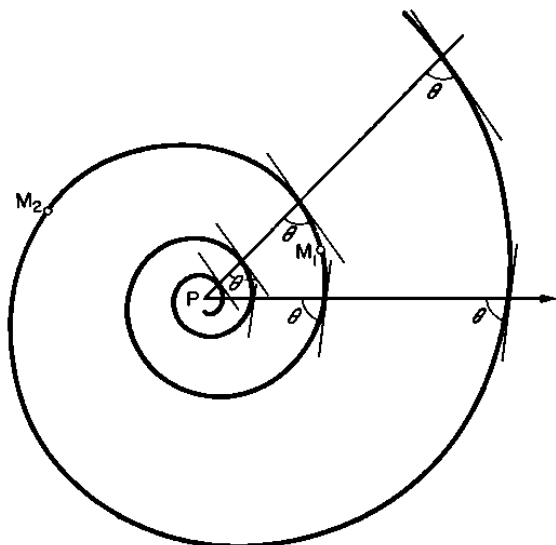


FIG. 67

from M_1 to M_2 will preserve a constant angle between the direction of motion and the direction to the pole. If we have a device indicating the direction to the pole, on knowing the angle θ (i.e. $\operatorname{arccot} m$) for the spiral through M_1 and M_2 , it can be verified that, having started to move from M_1 and moving along the plane in such a way that θ remains invariable, we necessarily arrive at M_2 . In this case the motion automatically proceeds along the arc of a logarithmic spiral.

Instrument flying is accomplished in a similar manner (without orientation about a locality). The device is the compass, the pole P is the earth's pole, and the role of logarithmic spiral on a plane is played on the terrestrial sphere by an analogous curve which cuts any meridian (i.e. direction to the pole) at a constant angle; this curve is also a spiral, drawn on a sphere, with the earth's pole as the point about which a point moving along the spiral and approaching the pole revolves indefinitely.

In general, a curve cutting a direction to a fixed point at a constant angle is called a loxodrome. Thus the loxodrome on a plane is a logarithmic spiral, whilst the corresponding curve on the terrestrial sphere is usually simply called a loxodrome. We therefore say where relevant that the flight is along a "loxodrome"; the constant angle between the direction of flight and the meridian is called the "path angle".

57. Rate of change of length of arc. We require for certain further applications of the derivative the concept of the rate of change of a length of arc, and in order to establish this we first of all need a definition of the general concept of length of arc. In essence, we must leave the consideration of this topic until Chapter VIII; we shall assume here that the idea of length of arc is obvious and shall also assume that our conception of length implies the following natural principle: *if convex step-lines are described about and inscribed in a convex* arc, the length of the arc is less than the length of the first step-line and greater than the length of the second.* As the reader will recall, this principle is used when evaluating the length of a circumference, and we used it for finding the limit $\sin \alpha / \alpha$ as $\alpha \rightarrow 0$ (Sec. 40).

Let a continuous curve, having a tangent at every point, be given in a system of Cartesian co-ordinates by the equation $y = f(x)$.

We take the arc MM' of the curve, corresponding to the interval $[x, x + \Delta x]$, and draw the tangent MT to it at its initial point M (Fig. 68). We shall assume that the arc is convex for sufficiently small Δx . This is usually true. Let Δs denote the length of arc $\widetilde{MM'}$; Δs is the increment of length s , measured from some point N .

By hypothesis,

$$MM' \leq \widetilde{MM'} = \Delta s \leq MT + TM'.$$

* An arc is said to be convex if it cuts any chord in only two points.

But

$$MM' = \sqrt{\Delta x^2 + \Delta y^2}, \quad MT = \sqrt{\Delta x^2 + dy^2} = \sqrt{1 + y'^2} \Delta x, \\ TM' = \varepsilon = dy - \Delta y$$

so that

$$\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x \leq \Delta s \leq \sqrt{1 + y'^2} \Delta x + \varepsilon.$$

On dividing by Δx (we assume for simplicity that $\Delta x > 0$), we get

$$\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \leq \frac{\Delta s}{\Delta x} \leq \sqrt{1 + y'^2} + \frac{\varepsilon}{\Delta x}.$$

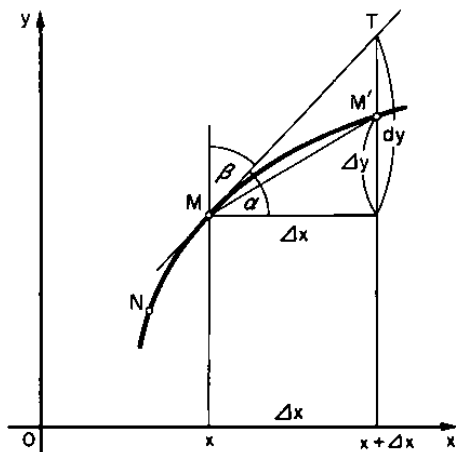


FIG. 68

The average rate of change of the length of arc therefore lies between two functions having the same limit $\sqrt{1 + y'^2}$, as $\Delta x \rightarrow 0$.

Hence

$$\frac{ds}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta s}{\Delta x} = \sqrt{1 + y'^2} \quad (= \sec \alpha, \text{ see Fig. 68}).$$

We find similarly that

$$\frac{ds}{dy} = \lim_{\Delta y \rightarrow 0} \frac{\Delta s}{\Delta y} = \sqrt{1 + x'^2} \quad (= \sec \beta, \text{ see Fig. 68}).$$

The rate of change of the length of arc with respect to either co-ordinate is measured by the secant of the angle between the tangent to the curve and the corresponding co-ordinate axis.

We have for the differential ds of the length of arc:

$$ds = \sqrt{1 + y'^2} dx = \sqrt{1 + x'^2} dy;$$

on taking dx (or dy) under the radical sign, we obtain a simple, easily memorized formula for the differential of the arc, which makes no previous assumption regarding the choice of independent variable:

$$ds = \sqrt{dx^2 + dy^2}.$$

GEOMETRICALLY: the differential of an arc can be represented by the length of the corresponding segment of the tangent to the arc at its initial point.

It follows from this that an arc of infinitesimal length differs from the corresponding segment of tangent by an infinitesimal of higher order. On combining this with the familiar property of the differential of the ordinate of a curve (Sec. 59), we can say that

A sufficiently small arc can be replaced, both as regards position and length, and with an arbitrarily small relative error, by the corresponding segment of the tangent at the initial point of the arc.

Notice the useful formulae:

$$\frac{dx}{ds} = \cos \alpha,$$

$$\frac{dy}{ds} = \cos \beta,$$

showing that dx and dy can be regarded as the projections of the segment of tangent, equal to ds , on Ox and Oy respectively.

The ratio of the length of arc MM' to the chord MM' tends to unity as the point M' tends to point M . For, as $\Delta x \rightarrow 0$,

$$\frac{\widetilde{MM'}}{MM'} = \frac{\Delta s}{\sqrt{\Delta x^2 + \Delta y^2}} = \frac{\frac{\Delta s}{\Delta x}}{\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}} \rightarrow \frac{\frac{ds}{dx}}{\sqrt{1 + (y')^2}} = 1.$$

We might have based our definition of the differential of an arc on a different principle to the above, namely, this last property of equivalence (in length) of an infinitesimal arc and its chord. It may easily be imagined that this property, as regards a circle, is expressed by our familiar limiting relationship $\lim_{\alpha \rightarrow 0} \frac{\sin \alpha}{\alpha} = 1$.

58. Processes of organic growth. The various processes studied in natural science are most commonly characterized by special features in the rate of change of the magnitudes taking part. We shall dwell here on a single example

of an important and wide-spread type of process in which *the rate of change of one magnitude, y , which is a function of another, x , is proportional to y itself*, i.e.

$$\frac{dy}{dx} = ky,$$

where k is a constant coefficient of proportionality.

We shall first take the easily visualized process of growth of a sum of money carrying compound interest. If the initial sum is A_0 roubles, and the compound interest is p per cent per annum, after t years (Sec. 22) the sum will have increased to $A = A_0(1 + p/100)^t$ roubles.

Now suppose that the annual rate of p percent is calculated every month instead of once a year; in other words, at the end of each month a fraction $p/12 \cdot 100$ is added to the capital. Obviously at the end of the first month the initial sum A_0 has increased to $A_0(1 + p/12 \cdot 100)$ roubles, and at the end of a year to $A_0(1 + p/12 \cdot 100)^{12}$ roubles, and after t years to the sum

$$A = A_0 \left(1 + \frac{k}{12}\right)^{12t}, \quad \text{where } k = \frac{p}{100}.$$

In the same situation we can work out the annual p percent every day, every hour, every minute and so on. In a word, suppose that the year is divided into n equal parts and at the end of each part we work out p/n per cent of the total initial capital at the time of the calculation. We find by arguing as above that the sum after t years is

$$A = A_0 \left(1 + \frac{k}{n}\right)^{nt}, \quad (\dagger)$$

where $k = p/100$.

We say that the annual p per cent is added continuously if the capital after t years is obtained from the formula got by passing to the limit as $n \rightarrow \infty$ in $(\dagger)$. We have:

$$A = \lim_{n \rightarrow \infty} A_0 \left(1 + \frac{k}{n}\right)^{nt} = A_0 \lim_{n \rightarrow \infty} \left[\left(1 + \frac{k}{n}\right)^n \right]^t = A_0 \left[\lim_{n \rightarrow \infty} \left(1 + \frac{k}{n}\right)^n \right]^t.$$

Consequently (Sec. 41):

$$A = A_0 e^{kt}.$$

In whatever way the compound interest is calculated, the sum A is given by an exponential function of t (time), though when the interest is added discontinuously the sense of the problem implies that the independent variable t can only take values which are multiples of $1/n$, and A changes by jumps, i.e. discontinuously; in the limiting case, when $A = A_0 e^{kt}$, i.e. with continuous addition, t can take any value and A always changes continuously*.

* It should not be thought that the capital increases much more rapidly when the interest is added continuously than when discontinuously (assuming the same p).

In fact, say 10 roubles at $p = 8\%$ increases after a year to 10.8 roubles (yearly addition), and to $10e^{0.08} \approx 10.83$ roubles (continuous addition); or 100 roubles at $p = 5\%$ gives after 20 years a sum of $100(1 + 1/20)^{20} \approx 266$ roubles (yearly addition) or $100e \approx 272$ roubles (continuous addition). The explanation of this situation is fairly obviously that, although a decrease in the interval

Let us find the rate of change of the sum $A = A_0 e^{kt}$ with respect to time t :

$$\frac{dA}{dt} = A_0 k e^{kt} \quad \text{or} \quad \frac{dA}{dt} = kA.$$

The coefficient of proportionality $k (= p/100)$ indicates the part of A (the sum) which is added to A after unit time t (after a year), on condition that the growth of A (the sum) is uniform during this unit time**.

Processes exactly similar to the above are found in many natural phenomena. For instance, in the growth of a simple organism (in the phenomenon of cell reproduction), the realization can be seen of a process of "continuous addition of compound interest". Each cell is a source of formation of new cells, so that the number of new cells generated per unit time is proportional to the initial quantity of cells. Assuming that the reproduction process is continuous (though strictly speaking it is discontinuous), the rate of growth of the organism at a given instant can be taken as proportional to its size (by weight or volume). Experiments support—within definite limits—the correctness of this assumption, together with all the mathematical consequences following from it.

Processes such as this are sometimes called processes of organic growth, and the condition that the rate of change of a magnitude be proportional to the magnitude itself is the law of organic growth.

We shall encounter later (Chapter VIII) several processes of great importance in science which evolve in accordance with the law of organic growth. They include such phenomena as the decay of radium, the heating of a body by heat transfer, the variation of atmospheric pressure, the course of a chemical reaction, etc.

5. Repeated Differentiation

59. Derivatives of higher orders. Let the function $y = f(x)$ have a derivative $f'(x)$ in some interval of variation of the independent variable x . The derivative of $f'(x)$ (if it exists) is termed the *second*

Continued from p. 187:

of time after which the interest is added implies an increase in the number of instants at which the addition is made, at the same time there is a decrease in the fraction of the existing sum formed by the added increment. The simultaneous operation of these two causes leads in the long run to an increase in the profit: $(1 + k/n)^n$ increases with increasing n , but relatively little in practice. Hence there is a preference for using the law of continuous addition. This law is very convenient in computations (there are simple tables of the values of e^{kt}), and theoretical discussions become simpler with continuous functions.

** This means that the continuous addition of interest refers only to the initial sum A (intermediate increments during the year are not taken into account), whilst this is equivalent to the simple addition at the end of the year of p percent to the sum of A roubles. For, the annual gain with continuous (but not compound) addition of interest will be equal to

$$\lim_{n \rightarrow \infty} \left(\frac{k}{n} A + \frac{k}{n} A + \dots + \frac{k}{n} A \right) = \lim_{n \rightarrow \infty} n \cdot \frac{k}{n} A = kA.$$

order derivative or *second derivative* of the original function $f(x)$ and is written as $f''(x)$:

$$f''(x) = (f'(x))' = \lim_{\Delta x \rightarrow 0} \frac{f'(x + \Delta x) - f'(x)}{\Delta x}.$$

Similarly, the derivative of the second derivative is called the *third order* or *third derivative* $f'''(x)$ of function $f(x)$. Let us give a general definition.

Definition. The n -th order or n -th derivative $f^{(n)}(x)$ is the derivative of the $(n - 1)$ -th derivative:

$$f^{(n)}(x) = (f^{(n-1)}(x))' = \lim_{\Delta x \rightarrow 0} \frac{f^{(n-1)}(x + \Delta x) - f^{(n-1)}(x)}{\Delta x}.$$

For the sake of uniform terminology, the derivative of the given function may be called the first derivative, and the function itself the derivative of zero order, $f(x) = f^{(0)}(x)$. Our ability to differentiate any elementary function enables us to find one after another the successive derivatives up to any order. It is sometimes possible simply to indicate the general form of the k -th order derivative for any k .

Examples. (1) $y = 3x^2 - 5x + 1$, $y' = 6x - 5$, $y'' = 6$, $y''' = 0$, in general $y^{(k)} = 0$ for all $k \geq 3$.

$$(2) \quad y = x^n, \quad y' = nx^{n-1}, \quad y'' = n(n-1)x^{n-2}, \dots, \quad y^{(k)} = n(n-1)\dots(n-k+1)x^{n-k};$$

if n is a positive integer, then

$$y^{(n)} = n! \quad \text{and} \quad y^{(n+1)} = y^{(n+2)} = \dots = 0.$$

$$(3) \quad y = \sin x, \quad y' = \cos x = \sin\left(x + \frac{\pi}{2}\right), \quad y'' = -\sin x = \sin\left(x + \pi\right), \dots \\ \dots, \quad y^{(k)} = \sin\left(x + k\frac{\pi}{2}\right).$$

$$(4) \quad y = e^x, \quad y^{(k)} = e^x.$$

$$(5) \quad y = a^x, \quad y' = a^x \ln a, \quad y'' = a^x (\ln a)^2, \dots, \quad y^{(k)} = a^x (\ln a)^k.$$

$$(6) \quad y = \ln(1+x), \quad y' = \frac{1}{1+x}, \quad y'' = -\frac{1}{(1+x)^2}, \dots$$

$$\dots, \quad y^{(k)} = \frac{(-1)(-2)\dots(-k+1)}{(1+x)^k} = (-1)^{k-1} \frac{(k-1)!}{(1+x)^k}, \quad k > 1.$$

If y is an implicit function, to find its higher derivatives we have to differentiate the equation defining it a corresponding number

of times, always remembering that y and all its derivatives are functions of the independent variable.

Thus we find the second derivative of the function y given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

by differentiating this equation twice. We obtain in turn

$$\frac{x}{a^2} + \frac{y}{b^2} y' = 0,$$

$$\left(\frac{x}{a^2}\right)' + \left(\frac{y}{b^2} \cdot y'\right)' = \frac{1}{a^2} + \frac{1}{b^2} y'^2 + \frac{y}{b^2} y'' = 0.$$

Hence

$$y'' = -\frac{b^2 + a^2 y'^2}{a^2 y}.$$

In this example we can easily express the function and its derivative explicitly in terms of x . We then have:

$$y'' = -\frac{ab}{\sqrt{(a^2 - x^2)^3}},$$

i.e. the same as is got by twice differentiating the explicit expression for the function. In general, the derivative of any order of an implicit function can always be expressed in the long run in terms of the independent variable and the function itself.

For instance, given the equation (see Secs. 13, 49)

$$xy - e^x + e^y = 0,$$

defining y implicitly as a function of x , differentiation twice in succession gives us:

$$y + xy' - e^x + e^y y' = 0,$$

and

$$y'' + y' + xy'' - e^x + e^y y'^2 + e^y y'' = 0.$$

On finding y' from the first equation and substituting in the second, we arrive at an equation from which y'' can be expressed in terms of x and y .

To find a higher order derivative of a function given parametrically, we have to differentiate the expression for the previous derivative as a function of the independent variable. Let $y = f(t)$, $x = \varphi(t)$.

We have

$$y' = \frac{dy}{dx} = \frac{f'(t)}{\varphi'(t)}.$$

Further,

$$y'' = \frac{d\left(\frac{f'(t)}{\varphi'(t)}\right)}{dx} = \frac{d\left(\frac{f'(t)}{\varphi'(t)}\right)}{dt} \frac{dt}{dx} = \frac{f''(t)\varphi'(t) - \varphi''(t)f'(t)}{\varphi'^2(t)} \frac{dt}{dx},$$

and since

$$\frac{dt}{dx} = \frac{1}{\varphi'(t)},$$

then

$$y'' = \frac{f''(t)\varphi'(t) - \varphi''(t)f'(t)}{\varphi'^3(t)}.$$

Differentiation of this expression with respect to x gives the third derivative and so on.

The derivatives of the second and higher orders are absolutely essential for defining certain important mathematical, mechanical and physical concepts, and for a fuller and subtler investigation of functions than is possible by using only the first derivative.

We take as an example a fundamental problem of mechanics, connected with Newton's basic laws. We shall confine ourselves for simplicity to the case of rectilinear motion under the action of a force acting along the same straight line. By Newton's second law, the increment in the velocity when the force is constant is proportional to the force and the increment of time, and inversely proportional to the mass. If the force is not constant, but is a function of time, to give expression to Newton's second law we have to turn to the concept of second derivative.

Let s denote the path traversed by the body moving in a straight line at the instant t ; then s' is the velocity at this instant. The magnitude s'' , measuring the rate of change of s' , is called the acceleration of the rectilinear motion at the instant t . Newton's second law can be written with the aid of s'' as: $ms'' = F(t)$, where m is the mass of the body, $F(t)$ is the force acting at the instant t .

The value of $f''(x)$ at a point x defines the rate of change of $f'(x)$ at this point, i.e. the rate of change of the rate of change of $f(x)$. If we borrow a term from mechanics, we can call $f''(x)$ the acceleration of the change of $f(x)$ at the given x .

The question of the geometrical meaning of the second derivative will be discussed in Chapter IV (Sec. 81).

A function may have derivatives up to a certain order at a given point, whilst higher order derivatives do not exist. However, every elementary function has in general (i.e. except at individual points) derivatives up to any order in its domain of definition.

60. Leibniz's formula. It is perfectly obvious that the rules for finding the derivative of a sum of functions and of a product of a constant and a function can be extended directly to higher order derivatives. In fact:

(1) *The n -th order derivative of a sum of a finite number of functions is equal to the sum of the n -th derivatives of the terms, i.e. if $y = u + v + \dots + w$, then*

$$y^{(n)} = u^{(n)} + v^{(n)} + \dots + w^{(n)}.$$

(2) *The n -th order derivative of the product of a constant and a function is equal to the product of the constant and the n -th order derivative of the function, i.e. if $y = cu$, where $c = \text{const}$, then*

$$y^{(n)} = cu^{(n)}.$$

As regards the other rules, we shall only mention a practically useful rule for finding the n -th derivative of a product of functions.

Let

$$y = uv;$$

we shall express $y^{(n)}$ in terms of the derivatives of u and v .

We have consecutively:

$$y' = u'v + uv',$$

$$y'' = u''v + 2u'v' + uv'',$$

$$y''' = u'''v + 3u''v' + 3u'v'' + uv'''. \quad \dots$$

An analogy may easily be seen between the expressions for the second and third derivatives and the binomial expansions to the second and third powers respectively: these expressions are constructed from the derivatives of u and v (of zero, first, second and third orders) in the same way as the binomial expansions from the successive powers of u and v (zero, first, second and third). This analogy happens to hold in the general case.

LEIBNIZ'S FORMULA. We have for any n :

$$y^{(n)} = (uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v' + \dots + \frac{n(n-1)}{2!}u^{(n-2)}v'' + \dots + nu'v^{(n-1)} + uv^{(n)}. \quad (\dagger)$$

This formula can be got from the binomial expansion of $(u + v)^n$ if the powers of u and v in it are replaced by the corresponding derivatives of u and v .

Proof. We shall prove (†) by passing from n to $n + 1$, i.e. by the so-called induction method. Suppose that (†) holds for n ; then we show that it also holds for $n + 1$. We differentiate equation (†) once:

$$\begin{aligned} y^{(n+1)} &= (uv)^{(n+1)} = u^{(n+1)}v + u^{(n)}v' + nu^{(n)}v' + nu^{(n-1)}v'' + \\ &\quad + \frac{n(n-1)}{2!}u^{(n-1)}v'' + \dots + uv^{(n+1)} \\ &= u^{(n+1)}v + (n+1)u^{(n)}v' + \frac{(n+1)n}{2!}u^{(n+1)}v'' + \dots + uv^{(n+1)}. \quad (\dagger\dagger) \end{aligned}$$

Let us show that (††) has the same form as (†).

In fact, let us write out only the $(k+1)$ -th and $(k+2)$ -th ($k < n$) terms on the right-hand side of (†):

$$C_n^k u^{(n-k)} v^{(k)} + C_n^{k+1} u^{(n-k-1)} v^{(k+1)},$$

where C_r^s denotes the number of combinations of r from s elements. On differentiating these terms, we obtain on the right-hand side of (††):

$$\begin{aligned} C_n^k u^{(n-k+1)} v^{(k)} + C_n^k u^{(n-k)} v^{(k+1)} + C_n^{k+1} u^{(n-k)} v^{(k+1)} + \\ + C_n^{k+1} u^{(n-k-1)} v^{(k+2)}; \end{aligned}$$

we group together the second and third of these terms:

$$(C_n^k + C_n^{k+1}) u^{(n-k)} v^{(k+1)},$$

which in fact yields the $(k+2)$ -th term in (††), since, as we know,

$$C_n^k + C_n^{k+1} = C_{n+1}^{k+1}.$$

On similarly grouping the first and last of the terms written above with the preceding and following terms respectively, we get the $(k+1)$ -th and $(k+3)$ -th terms in (††), which are constructed similarly:

$$C_{n+1}^k u^{(n-k+1)} v^{(k)} \quad \text{and} \quad C_{n+1}^{k+2} u^{(n-k-1)} v^{(k+2)}$$

and so on.

Thus formula (††) is got from (†) by replacing n by $n + 1$, which is what we wished to show; since formula (†) in fact holds, as we have seen, for $n = 2$ (and $n = 3$), it follows from this and the proposition that we have just proved that it holds for any n .

Leibniz's formula is often very useful, since, by using the successive derivatives of two factors, we are enabled to form at once the expression for the corresponding derivative of their product.

A formula can similarly be deduced for the n -th order derivative of the product of several factors $uv \dots w$; it is similar to the formula for the expansion of $(u + v + \dots + w)^n$.

Example. Let us find the hundredth derivative of the function $y = x^2 \sin x$.

We have:

$$\begin{aligned} y^{(100)} &= (x^2 \sin x)^{(100)} = \\ &= (\sin x)^{(100)} x^2 + 100 (\sin x)^{(99)} (x^2)' + \frac{100 \cdot 99}{2} (\sin x)^{98} (x^2)''; \end{aligned}$$

there is no need to work out the following terms since they are all zero: each contains as a factor a derivative of x^2 of higher than the second order. Hence (see Sec. 59),

$$\begin{aligned} x^{(100)} &= x^2 \sin \left(x + 100 \frac{\pi}{2} \right) + 200 x \sin \left(x + 99 \frac{\pi}{2} \right) + \\ &+ 9900 \sin \left(x + 98 \frac{\pi}{2} \right) = x^2 \sin x - 200 x \cos x - 9900 \sin x. \end{aligned}$$

61. Differentials of higher orders. The differential dy of the function $y = f(x)$ is a function of two variables (Sec. 52): the independent variable x and its differential dx . The differential dx of the independent variable x is regarded as a magnitude independent of x : we can quote values of dx without taking account of the actual value of x :

Let us take the differential $d(df(x))$ of $df(x)$ as a function of x , i.e. the principal part of the increment

$$df(x + dx) - df(x),$$

proportional to dx . If this differential exists, it is called the second order or *second differential* of function $f(x)$ and is written as d^2y :

$$d^2y = d(dy).$$

Similarly, the *third differential* d^3y of $f(x)$ is the differential of the second differential as a function of x . Let us give a general definition.

Definition. The n -th differential $d^n y$ is the differential of the $(n - 1)$ -th differential as a function of x :

$$d^n y = d(d^{n-1} y).$$

Let us now find an expression for the higher order differentials of function f , on the assumption that its argument is the independent variable x . We have for the second differential:

$$d(dy) = (dy)' dx = [f'(x) dx]' dx,$$

and since, by what has been said, dx must be regarded as a constant when differentiating with respect to x , we have

$$d^2y = f''(x) dx \cdot dx = f''(x) dx^2.$$

Similarly, we have in general for the n -th differential:

$$d^n y = (d^{n-1} y)' dx = [f^{(n-1)}(x) dx^{n-1}]' dx = f^{(n)}(x) dx^n,$$

where dx^n denotes the n -th power of dx .

Thus the n -th differential is equal to the n -th derivative with respect to the independent variable multiplied by the n -th power of the differential of the independent variable.

The differential $df(x)$ of function $f(x)$ is called, for the sake of uniform terminology, the *first* (or first order) *differential*.

Each differential is an infinitesimal of higher order than all the lower order differentials:

$$\frac{d^n y}{d^m y} = \frac{f^{(n)}(x) dx^n}{f^{(m)}(x) dx^m} = \frac{f^{(n)}(x)}{f^{(m)}(x)} dx^{n-m} \rightarrow 0$$

as $dx \rightarrow 0$ if $n > m$ and $f^{(m)}(x) \neq 0$. On taking dx or dy (assuming $f'(x) \neq 0$) as the infinitesimal with which all the rest are compared, it will be seen that the n -th differential (if it is not equal to zero) is an infinitesimal of the n -th order (Sec. 39). The successive differentials

$$dy, d^2y, d^3y, \dots, d^ny, \dots$$

are arranged in order of increasing smallness.

If $y = f(x)$, then $dy = f'(x) dx$, independently of whether the argument x is the independent variable or a function of it (the property of invariance of the form of the first differential, see Sec. 52).

The differentials of higher orders no longer possess this property.

Suppose that x is not the independent variable, as we have so far assumed, but is a function of it. In this case, dx will depend on the independent variable and can no longer be regarded as a constant when differentiating the first differential with respect to the independent variable; a different expression to the one above is obtained for d^2y . In fact, we obtain on taking the differential of dy , by the rule for differentiation of a product:

$$d^2y = d[f'(x) dx] = d[f'(x)] dx + d(dx) f'(x).$$

But

$$d[f'(x)] = f''(x) dx \quad \text{and} \quad d(dx) = d^2x;$$

therefore

$$d^2y = f''(x) dx^2 + f'(x) d^2x.$$

We see that an extra term $f'(x) dx^2$ makes its appearance. It vanishes if x is the independent variable:

$$d^2x = (x)'' dx^2 = 0 \cdot dx^2 = 0.$$

The third order differential has a still more complicated form:

$$d^3y = f'''(x) dx^3 + 3f''(x) dx d^2x + f'(x) d^3x.$$

Notice that each term in the expression for the n -th differential (if it does not vanish) must be an n -th order infinitesimal with respect to the differential of the independent variable.

Thus, when finding the higher order differentials, we must pay careful attention to the nature of the argument of the function to be differentiated: is it or is it not the independent variable?

The formulae for the differentials lead us to expressions for the derivatives in the form of fractions:

$$\begin{aligned} y' &= f'(x) = \frac{dy}{dx}, \\ y'' &= f''(x) = \frac{d^2y}{dx^2}, \\ &\dots \dots \dots \\ y^{(n)} &= f^{(n)}(x) = \frac{d^ny}{dx^n}. \end{aligned}$$

These formulae—apart from the first, which is always valid—only hold when x is the independent variable.

As a matter of convenience, we often write $\frac{d^n}{dx^n}y$ instead of $\frac{d^ny}{dx^n}$. For example, we can write: $\frac{d^3}{dx^3}(2x^4 - x + 1) = 48x$.

CHAPTER IV

THE INVESTIGATION OF FUNCTIONS AND CURVES

1. The Behaviour of a Function "at a Point"

62. Construction of a graph from "elements". Drawing the graph of a function by plotting points corresponding to arbitrarily chosen points on the axis of the independent variable can yield a curve that deviates substantially from the true graph. A knowledge of the numerical derivatives enables us to make the drawing precise. For, if we work out the values of the function and its derivative for a given value of the independent variable, we can indicate not only the corresponding point of the graph but also the direction which the curve must take at this point.

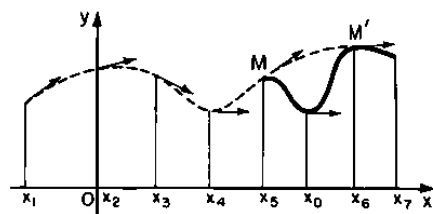


FIG. 69

The set of two numbers: the value of the function and the value of its derivative, is called the element of the function for the given value of the independent variable. Graphically, the element of a function is represented by a point on the plane and a vector starting from this point, the slope of the vector being equal to the numerical derivative.

Having found the elements of a function, we can stick to the directions which are indicated by these vectors when drawing the curve joining the points (Fig. 69). If one of them differs substantially from the straight direction to the next point of the graph, this shows that the function cannot be represented in the relevant

interval by a smooth curve close to the straight segment joining the two points..

Whilst we can obtain a more accurate graph by working out elements than by simply plotting points, we can still not guarantee that there will be no serious errors. For instance, the graph may in fact start off from the point M (Fig. 69) in the direction MM' , but then may bend away acutely, and the bend will not be discovered. Yet it is precisely this bend which indicates a change in the nature of the variation of the function. We can obviously avoid all such errors if we have advance knowledge of the values of the independent variable for which the function changes the nature of its variation. In Fig. 70, the function $y = f(x)$, represented by the graph AB , changes the nature of its variation at the points x_1, x_2, x_3, x_4 . The function is monotonic in an interval between two such consecutive points, i.e. it is either only increasing or only decreasing (see Sec. 15, IV).

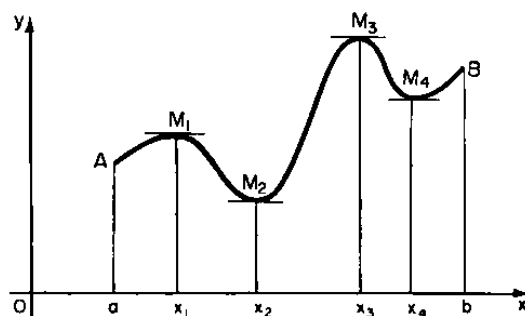


FIG. 70

Therefore, if we find in advance the points at which the function changes the nature of its variation, we no longer need to choose at random the points on the axis of the independent variable and draw the graph blindfold; instead, by marking off these special points, we thereby divide the entire interval into intervals in which the function is monotonic, and in which its behaviour cannot present any great surprises.

It is vitally important that the points mentioned, and the intervals in which the function is monotonic, can be found by considering the sign of the derivative. This close connection between the nature of the variation of the function and the sign of its derivative plays a large part in problems concerned with the investigation of functions.

63. Behaviour of a function “at a point”. Extrema. Before turning to a study of the behaviour of a function in an interval, we shall consider its behaviour in the neighbourhood of an individual point, or, to put the matter briefly, “at a point”.

Definition 1. A function $f(x)$ is increasing “at a point” x_0 if there exists an ε -neighbourhood of x_0 such that $f(x_0)$ is greater than $f(x)$ when x lies in the neighbourhood to the left of x_0 , and less than $f(x)$ when x is in the neighbourhood to the right of x_0 .

Definition 2. A function $f(x)$ is decreasing “at a point” x_0 if there exists an ε -neighbourhood of x_0 such that $f(x_0)$ is less than $f(x)$ when x is to the left of x_0 in the neighbourhood, and is greater than $f(x)$ when x is to the right of x_0 in the neighbourhood,

Definition 3. A point x_0 is called a maximum point of a function $f(x)$ if there exists an ε -neighbourhood of x_0 such that $f(x_0)$ is greater than $f(x)$ when x belongs to this neighbourhood ($f(x_0)$ is the greatest value of $f(x)$ in this neighbourhood).

Definition 4. A point x_0 is called a minimum point of $f(x)$ if there exists an ε -neighbourhood of x_0 such that $f(x_0)$ is less than $f(x)$ when x belongs to this neighbourhood ($f(x_0)$ is the least value of $f(x)$ in this neighbourhood).

As usual, we shall picture the independent variable x as a point moving in the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ and passing from left to right through the point x_0 ; if it is permissible to describe the situation in more visual (though not strictly accurate) terms, we can say that the above four cases imply respectively that the function:

- (1) “passes from smaller to larger values” (the function is increasing “at the point” x_0),
- (2) “passes from larger to smaller values” (the function is decreasing “at the point” x_0),
- (3) “passes from smaller values back to smaller values” (the function has a maximum “at the point” x_0),
- (4) “passes from larger values back to larger values” (the function has a minimum “at the point” x_0).

The differences in the nature of the graph in the neighbourhood of the point x_0 in our four cases is shown in Fig. 71. The arc M_1M_2 of the graph corresponding to the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ has the property that: in case (1) it lies below the straight line $y = y_0$ to the left of the point $M_0(x_0, y_0 = f(x_0))$, and above it to the right; in

case (2) it lies above $y = y_0$ to the left of M_0 , and below it to the right; in case (3) it always lies below $y = y_0$, and in case (4) it always lies above $y = y_0$.

The maximum and minimum have the common name of extremum. If the function $f(x)$ has an extremum at the point x_0 , the value $f(x_0)$ is called an extremal value—either maximum or minimum—, whilst

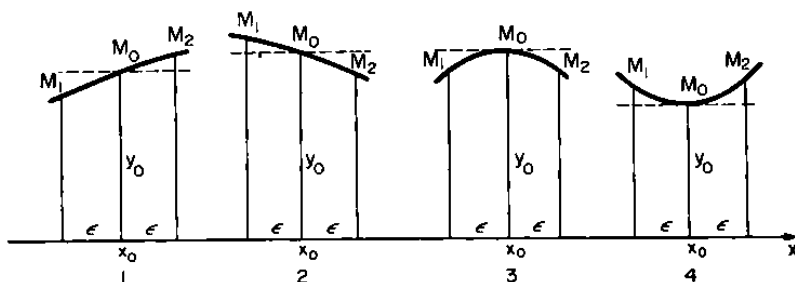


FIG. 71

x_0 is the extremal point. We now see that the points about which we spoke in Sec. 62—in which the function changes the nature of its variation—are in fact the extremal points of the function.

Let us give our four definitions with the aid of inequalities:

(1) The function $y = f(x)$ is increasing “at the point” x_0 if an ϵ exists such that

$$f(x_0 - \Delta x) < f(x_0) < f(x_0 + \Delta x)$$

for any Δx satisfying the condition $0 < \Delta x < \epsilon$.

(2) $y = f(x)$ is decreasing “at the point” x_0 if there exists an ϵ such that

$$f(x_0 - \Delta x) > f(x_0) > f(x_0 + \Delta x)$$

for any Δx satisfying $0 < \Delta x < \epsilon$.

(3) The point x_0 is called a maximum point of $y = f(x)$ if an ϵ exists such that

$$f(x_0 - \Delta x) < f(x_0), \quad f(x_0) > f(x_0 + \Delta x)$$

for any Δx satisfying $0 < \Delta x < \epsilon$.

(4) x_0 is called a minimum point of $y = f(x)$ if an ϵ exists such that

$$f(x_0 - \Delta x) > f(x_0), \quad f(x_0) < f(x_0 + \Delta x)$$

for any Δx satisfying $0 < \Delta x < \epsilon$.

It would be possible to provide in the above definitions for the possibility of $f(x)$ taking values equal to $f(x_0)$ in the ε -neighbourhood of x_0 at points lying to the left or to the right of x_0 . It would then be necessary to replace the signs of strict inequality in all the above inequalities by the signs of "non-strict" inequality ($<$ by $\leq$ and $>$ by $\geq$) and to suitably modify the wording. If it is in fact the case that $f(x) = f(x_0)$ at all points of the interval $[x_0 - \varepsilon, x_0]$ or of $[x_0, x_0 + \varepsilon]$, this case could be included as desired either in one of the first two cases or in one of the last two. Suppose, for instance, that $f(x) = f(x_0)$ for $x_0 - \varepsilon \leq x \leq x_0$ and $f(x) > f(x_0)$ for $x_0 < x < x_0 + \varepsilon$; in the conditions of non-strict inequalities, $f(x)$ can be regarded either as increasing at x_0 or as attaining a minimum at x_0 .

However, we shall not modify our definitions but shall leave the cases just mentioned (of $f(x) = f(x_0)$, as also, of course, the case of $f(x) = f(x_0)$ throughout the interval $[x_0 - \varepsilon, x_0 + \varepsilon]$) as unclassified.

This is all the more appropriate in that our above classification of functions according to their behaviour at a point (even in the conditions of non-strict inequalities) is not in any case exhaustive. For instance, if we take the familiar function $y = x \sin \frac{1}{x}$ at $x \neq 0$ and $y = 0$ at $x = 0$, we cannot say that it is increasing at $x = 0$, or decreasing, or that it has an extremum there. In fact, the function takes positive as well as negative values in any neighbourhood of $x = 0$, i.e. values both greater and less than its value at $x = 0$. At this point the function has a special feature: however small the interval with an end at $x = 0$ (i.e. either $[-\varepsilon, 0]$ or $[0, +\varepsilon]$ with any positive ε), it is not an interval of monotonicity of the function. In general, a function defined in a neighbourhood of a point x_0 (excepting possibly x_0 itself), and possessing the feature that no interval starting or ending at x_0 is an interval of monotonicity, will be described by us as "infinitely oscillating at x_0 ". The graph of an "infinitely oscillating" function at x_0 is represented in a neighbourhood of x_0 by a wavy curve consisting of an infinity of "waves", which naturally cannot be fully illustrated in a figure.

If a function is defined in a given closed interval and is not "infinitely oscillating", every point of the interval (including its end) is the end and beginning of intervals in which the function is monotonic or constant.*

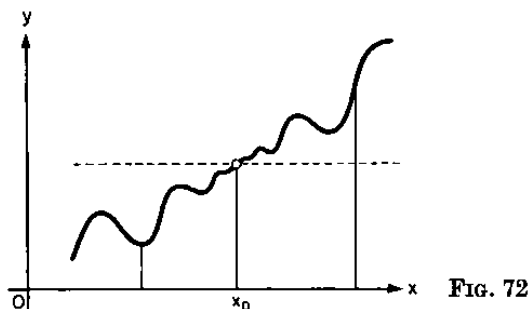
Such a function belongs at any point of the interval to one of the four types established above (with the conditions of non-strict inequalities): it is either increasing or decreasing or has a maximum or a minimum.

It should be borne in mind that the converse does not hold: a function may be of one of the above four types yet still be "infinitely oscillating". It is easy to imagine, at any rate graphically, a function which is increasing (or decreasing, or possessing an extremum) at the point x_0 and which is "infinitely oscillating" at $x = x_0$ (Fig. 72).

If a function is not "infinitely oscillating" in a given finite interval, it has a finite number of extrema in this interval. The converse is also true.

A function which is "infinitely oscillating at a point x_0 " has an infinite number of extrema in any neighbourhood of x_0 .

* Obviously, the left-hand end of the original interval can only be the beginning, and the right-hand end the end, of an interval in which the function is constant or monotonic.



We are not concerned in practice with “infinitely oscillating” functions. Let us remark that no elementary function can be “infinitely oscillating” in any closed interval in which it is defined.

Like the concepts of increase and decrease of a function at a point, the concepts of maximum and minimum values of the function—as distinct from its greatest and least values (or absolute maximum and minimum) in an interval—have a “local” character: they are based on a comparison of the given value of the function with its values at all points of a sufficiently small neighbourhood. Thus, for instance, a maximum may be, in the interval as a whole, less than a minimum of the function. In Fig. 70, $f(x)$ has a maximum $f(x_1)$ which is less than the minimum $f(x_4)$.

It may also be remarked that, by definition, an extremal point necessarily lies inside the domain of definition of the function.

64. Tests for the behaviour of a function “at a point”. The behaviour of a function “at a point” is very closely connected with the numerical derivative. This connection is based on theorems which are proved with the aid of simple propositions from the theory of limits. We shall start by giving these propositions.

THEOREM. Let the function $u(\alpha)$ tend to a limit a as $\alpha \rightarrow \alpha_0$:

$$\lim_{\alpha \rightarrow \alpha_0} u(\alpha) = a;$$

(1) if $u(\alpha) > 0$ for all α in a neighbourhood of the point α_0 , then $a \geq 0$;

(2) conversely, if we know that $a > 0$, then $u(\alpha) > 0$ in a neighbourhood of α_0^* .

* This theorem leads to the following more general theorem: if

$$\lim_{\alpha \rightarrow \alpha_0} u(\alpha) = a, \quad \lim_{\alpha \rightarrow \alpha_0} v(\alpha) = b$$

and $u(\alpha) > v(\alpha)$ for all α in a neighbourhood of α_0 , then $a \geq b$. Conversely, if we know in advance that $a > b$, we have $u(\alpha) > v(\alpha)$ in a neighbourhood of α_0 . To prove this, we only need to consider the difference $u(\alpha) - v(\alpha)$.

Proof.

(1) The function $u(\alpha)$ cannot tend to a negative number, since in this case its values in a sufficiently small neighbourhood of α_0 would differ by as little as desired from this negative number and would therefore themselves be negative. However, a positive function can tend to zero.

(2) In a sufficiently small neighbourhood of the point α_0 , $u(\alpha)$ differs by as little as desired from its limit a , which is a positive number, i.e. the function is positive in this neighbourhood.

We now turn to two theorems on the connection between the local behaviour of a function and the numerical derivatives.

DIRECT THEOREM. If a function $f(x)$ is increasing "at a point" x_0 and is differentiable for $x = x_0$, then

$$f'(x_0) \geq 0.$$

If a function $f(x)$ is decreasing "at a point" x_0 and is differentiable for $x = x_0$, then

$$f'(x_0) \leq 0.$$

Proof. If the function is increasing "at the point" x_0 , $\Delta y = f(x_0 + \Delta x) - f(x_0)$ is positive for sufficiently small positive Δx and negative for sufficiently small negative Δx . Hence the ratio $\Delta y/\Delta x$ is positive in a neighbourhood of x_0 , i.e. tends to a non-negative number. It follows from this that $f'(x_0) \geq 0$.

Similarly, if the function is decreasing at x_0 , the ratio $\Delta y/\Delta x$ is negative in a neighbourhood of x_0 , i.e. $f'(x_0) \leq 0$.

Some examples will show that the sign of equality may in fact hold. Thus the function $y = x^3$ is increasing at every point, in particular at $x = 0$ (the function is negative to the left of $x = 0$ and positive to the right), whilst its derivative $(x^3)' = 3x^2$ vanishes at $x = 0$.

This theorem implies geometrically that the tangent to the curve $y = f(x)$ at the point $M_0(x_0, f(x_0))$ is inclined to Ox at an acute or zero angle when $f(x)$ is increasing at the point x_0 , and at an obtuse or zero angle when decreasing.

The converse theorem is of more value.

CONVERSE. If $f'(x_0) > 0$, $f(x)$ is increasing "at the point" x_0 . If $f'(x_0) < 0$, $f(x)$ is decreasing at x_0 .

Proof. If $f'(x_0) > 0$, the ratio

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is positive for sufficiently small Δx , i.e. $f(x_0 + \Delta x) - f(x_0) > 0$ for $\Delta x > 0$ and $f(x_0 + \Delta x) - f(x_0) < 0$ for $\Delta x < 0$, i.e. $f(x_0)$ is less than the values taken by $f(x)$ to the right of x_0 , and greater than the values to the left, i.e. $f(x)$ is increasing.

The second part of the theorem is proved similarly.

Hence, if $f'(x_0) \neq 0$, as the independent variable passes through the point x_0 the function "passes from smaller to larger values", or conversely.

If $f'(x_0) = 0$, $y = f(x)$ can vary in quite diverse ways, as examples have shown, when the independent variable passes through x_0 : it may be increasing at x_0 , or decreasing, or have an extremum at $x = x_0$. For instance, the function $y = x^3$ is increasing at $x = 0$, the function $y = -x^3$ decreasing; the function $y = x^2$ has a minimum, $y = -x^2$ has a maximum, whilst the derivatives of both vanish at $x = 0$.

If $f'(x_0) = 0$, the function $f(x)$ may also be "infinitely oscillating" at the point $x = x_0$. For instance, the function given by: $y = x^2 \sin \frac{1}{x}$ for $x \neq 0$, $y = 0$ for $x = 0$, is "infinitely oscillating" at $x = 0$ (Fig. 73), whilst its derivative is zero at $x = 0$.

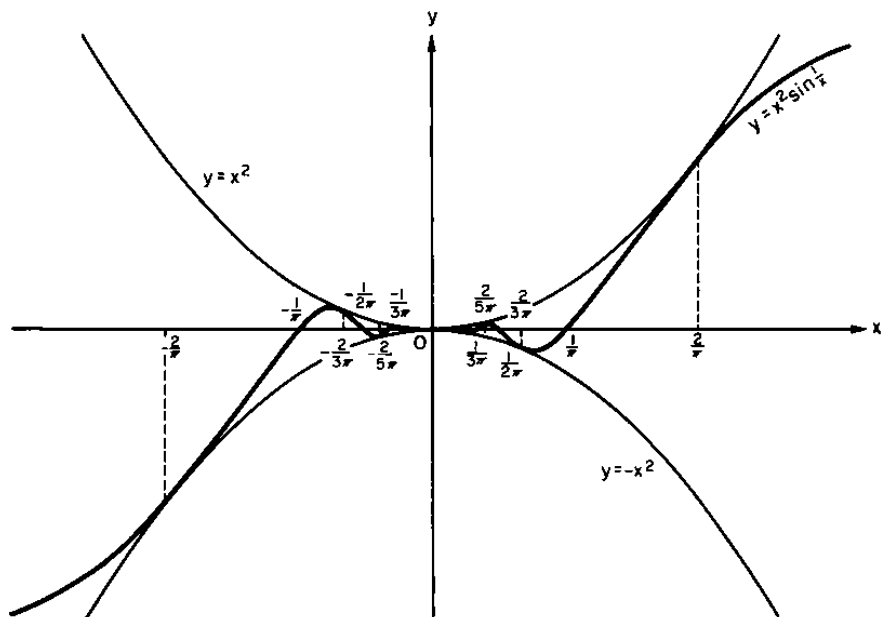


FIG. 73

It may be mentioned that, for all functions which are not "infinitely oscillating", i.e. for all functions ordinarily encountered, *definiteness of the sign of the derivative at a point guarantees the existence of a neighbourhood of the point in which the function is monotonic.*

Whatever the circumstances, *a point at which the derivative vanishes is called a stationary point of the function* (its rate of change is zero).

NECESSARY TEST FOR AN EXTREMUM. If a function $f(x)$ is differentiable at a point x_0 and has an extremum there, $f'(x_0) = 0$.

Proof. If $f'(x_0)$ were different from zero, the function $y = f(x)$ would be either increasing or decreasing at x_0 , by what has been proved, i.e. it would not have an extremum there.

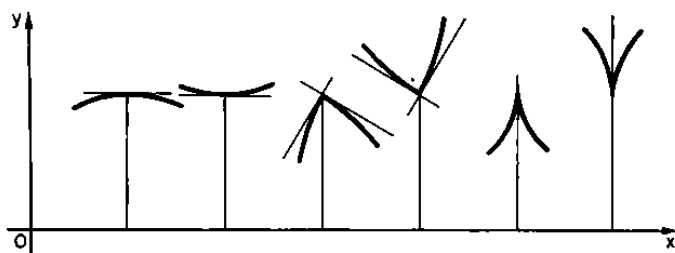


FIG. 74

This theorem has a simple geometrical interpretation: the tangent (if it exists) at a point of the curve $y = f(x)$ corresponding to an extremum must be parallel to Ox (Fig. 74). But a function may have an extremum at individual points at which it is non-differentiable (Fig. 74).

Thus *the extremal points of a function can only be points at which the derivative vanishes (i.e. stationary points) or does not exist.*

We can judge the behaviour of a function "at a point" from its numerical derivative. To see how a function varies throughout a given interval, we have to turn to the derivative itself (the "derived function").

2. Applications of the First Derivative

65. Theorems of Rolle and Lagrange.

ROLLE'S THEOREM*. If the function $f(x)$ is continuous in a closed interval $[x_1, x_2]$, is differentiable in the open interval (x_1, x_2) and has

* M. Rolle was a French mathematician (1652–1719).

equal values at the ends of the interval, there exists at least one value $x = \xi$ in this interval for which $f'(\xi) = 0$.

Proof. If the values of the function are equal at the ends of the interval, $f(x_1) = f(x_2)$, then either the function does not change at all inside the interval, and always $f(x) = f(x_1) = f(x_2)$, or else it does vary; if it varies, it attains a greatest (as in Fig. 75) or a least value, this value being evidently inside the interval $[x_1, x_2]$, i.e.

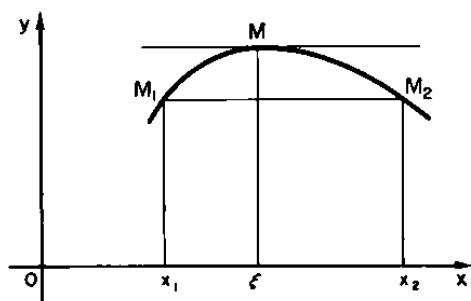


FIG. 75

the function has an extremum inside the interval. In the first case the derivative is zero for all values of x , in the second, by what was proved in Sec. 64, it is zero at the extremal points*.

Rolle's theorem can be interpreted geometrically as: *there is a point on the arc of the curve $y = f(x)$ at which the tangent is parallel to the axis of abscissae* (Fig. 75).

If, in particular, $f(x_1) = f(x_2) = 0$, Rolle's theorem can be stated as follows:

At least one zero of the derivative lies between any two zeroes of a function (see Sec. 15).

It was actually in this form, and as applied only to polynomials, that the theorem was first stated by Rolle. It has found many applications in analysis and algebra.

Rolle's theorem is a particular case of the following.

LAGRANGE'S THEOREM.** If a function $f(x)$ is continuous in a closed interval $[x_1, x_2]$ and is differentiable in the open interval (x_1, x_2) ,

* These may not be extrema in the "strict" sense—nevertheless the derivative vanishes at the corresponding point (because otherwise the function would be increasing or decreasing at this point).

** Lagrange (1736–1813) was a famous French mathematician.

there exists in this interval at least one value $x = \xi$ for which

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi).$$

In other words, the average rate of change of the function $f(x)$ in the interval $[x_1, x_2]$ is equal to its rate of change at some intermediate "average" point ξ ($x_1 < \xi < x_2$ or $x_1 > \xi > x_2$).

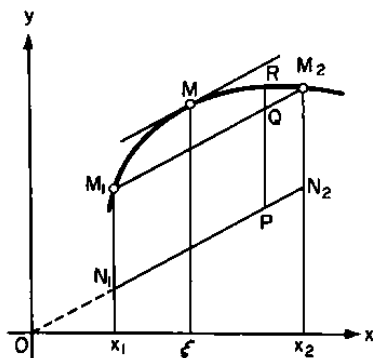


FIG. 76

Proof. We reduce the proof of Lagrange's theorem to Rolle's theorem. We subtract from the given function $y = f(x)$ a linear function $y = \lambda x$, the graph of which is the straight line ON_2 , parallel to the chord M_1M_2 of the graph of $y = f(x)$ (Fig. 76). The value of our auxiliary function,

$$y = f(x) - \lambda x,$$

will be represented by the segment PR , equal to the sum of the constant segment $PQ = N_1M_1 = N_2M_2$ and segment QR . Thus the chord joining the end-points of the arc corresponding to the new function will be parallel to Ox .

We thus take the auxiliary function

$$F(x) = f(x) - \lambda x,$$

where λ is the slope of the straight line N_1N_2 , equal to the slope of the chord M_1M_2 :

$$\lambda = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The function $F(x)$ satisfies all the conditions of Rolle's theorem: it is continuous in the interval $[x_1, x_2]$ and differentiable in (x_1, x_2) ,

since both $f(x)$ and λx possess these properties, whilst its values at the ends of the interval are equal*:

$$F(x_1) = F(x_2).$$

By Rolle's theorem, there exists in the interval (x_1, x_2) a point ξ such that $F'(\xi) = 0$. But $F'(x) = f'(x) - \lambda$; hence $f'(\xi) - \lambda = 0$, whence

$$\lambda = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi).$$

This is what we had to prove.

The geometrical meaning of Lagrange's theorem is that *there is at least one point on the arc of the curve $y = f(x)$ at which the tangent is parallel to the chord*.

Rolle's theorem is obtained from Lagrange's theorem in the particular case when $f(x_1) = f(x_2)$.

The theorems of Rolle and Lagrange cease to be true if we do not require differentiability of the function at every point of the interval (x_1, x_2) . For instance, the continuous function $y = 3 - \sqrt[3]{x^2}$ has equal values ($= 2$) at the ends of the interval $[-1, +1]$ whilst at the same time its derivative $y' = -2/(3\sqrt[3]{x})$ vanishes nowhere. In fact, the conditions of the theorem are not fulfilled in this case: the derivative does not exist at the interior point $x = 0$ of the interval.

We have by Lagrange's theorem:

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(\xi), \quad x_1 < \xi < x_2 \quad (*)$$

This formula enables us to state the theorem as:

The increment of a function is equal to the increment of the independent variable multiplied by the derivative at some "mean" point.

Lagrange's theorem is also known as the "mean value theorem of the differential calculus" or "the theorem on finite increments."

Formula (*) is known as Lagrange's formula (or the formula of finite increments). It enables us to give an accurate expression for the increment of a function in terms of the values of the derivative at some point and the increment of the argument, without any

* This is also easily proved directly. In fact:

$$\begin{aligned} F(x_1) &= f(x_1) - \frac{f(x_2) - f(x_1)}{x_2 - x_1} x_1 = \frac{x_2 f(x_1) - x_1 f(x_2)}{x_2 - x_1}, \\ F(x_2) &= f(x_2) - \frac{f(x_2) - f(x_1)}{x_2 - x_1} x_2 = \frac{x_2 f(x_1) - x_1 f(x_2)}{x_2 - x_1}. \end{aligned}$$

assumptions regarding the size of this increment. Whilst of great theoretical value, Lagrange's theorem (and formula) are in themselves of little use for calculations, since the theorem asserts only the existence of the number ξ satisfying equation(*), and gives no indication as to how it may actually be found.

In a few cases, however, we know in advance what ξ must be. For instance, for linear and quadratic functions the point ξ is always the mean of points x_1 and x_2 , i.e. $\xi = \frac{1}{2}(x_1 + x_2)$ (the reader will easily prove this). In other cases, the position of the point ξ depends on the function $f(x)$ and the interval $[x_1, x_2]$.

Lagrange's formula can be rewritten as

$$\Delta f(x) = f(x + \Delta x) - f(x) = f'(\xi)\Delta x$$

or, bearing in mind that ξ is intermediate between the points x and $x + \Delta x$, as

$$\Delta f(x) = f'(x + \theta\Delta x)\Delta x,$$

where θ is a positive number less than unity: $0 < \theta < 1$.

This is a common form of Lagrange's formula.

Let us compare the formula of finite increments, written as

$$f(x_0 + \Delta x) = f(x_0) + f'(\xi)\Delta x, \quad \xi = x_0 + \theta\Delta x \quad (*)$$

with the formula

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x, \quad (**)$$

which follows from the expression for the differential (Sec. 53); we could, by the way, call(**) the "formula of infinitesimal increments".

With correct choice of ξ , (*) is accurate for any Δx , but is unsuitable for calculating the values of the function (ξ is unknown); (**) is suitable for calculating the values of the function (it is sufficient to know the function and its derivative at one point x_0), but is not accurate and in general only has a meaning for sufficiently small Δx .

66. Application of Lagrange's formula to approximations. If we take $\xi = x_0 + \theta(x_1 - x_0)$, where θ is a fully defined positive number less than unity, for all values $x = x_0 + \Delta x$ in an interval $[x_0, x_1]$, Lagrange's formula(*) becomes an approximate formula for calculating the values of the function in the interval $[x_0, x_1]$. With $\theta = 0$ this approximate formula is none other than differential formula (**) of Sec. 65. With $\theta = \frac{1}{2}$, for instance, we get the approximate formula:

$$f(x_0 + \Delta x) \approx f(x_0) + f'\left(\frac{x_0 + x_1}{2}\right)\Delta x$$

(which may be written alternatively as

$$f(x) \approx f(x_0) + f' \left(\frac{x_0 + x_1}{2} \right) (x - x_0).$$

Replacing $f(x_0 + \Delta x)$ by the linear expression on the right implies geometrically replacing the arc of the graph M_1M_2 (Fig. 77) by the piece M_1Q of the straight line parallel to the tangent to the curve at M , with abscissa $\frac{1}{2}(x_0 + x_1)$. It is immediately clear from the figure that, apart from points very close to x_0 , this substitution can

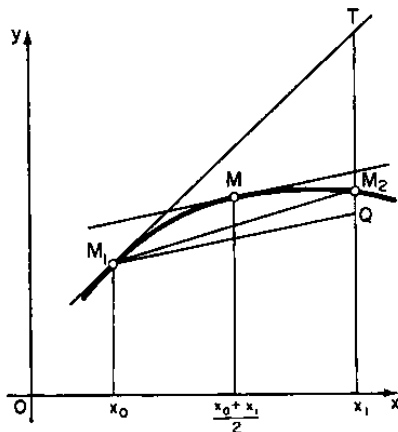


FIG. 77

be more satisfactory than replacing the arc M_1M_2 by the segment of tangent M_1T , i.e. for approximating the values of the function in the entire interval with Lagrange's formula, it is generally more satisfactory to take $\theta = \frac{1}{2}$ than $\theta = 0$ in the formula.

Let us calculate e.g. $\ln 782$ and $\ln 783$, given that $\ln 781 = 6.66058$ (see Sec. 53). We have $x_0 = 781$ and take $x_1 = 789$; we now have, on taking $\theta = \frac{1}{2}$ in this interval:

$$\ln 782 = \ln 781 + \frac{1}{185} \cdot 1 \approx 6.66058 + 0.00127 = 6.66185,$$

$$\ln 783 = \ln 781 + \frac{1}{185} \cdot 2 \approx 6.66058 + 0.00254 = 6.66312.$$

We find respectively from five-figure tables 6.66185; 6.66313. It is easy to see why our approximations above are on the low side.

Lagrange's formula enables us to estimate an error, i.e. to find the maximum error ε in calculating the values of $y = f(x)$ in the interval $[x_0, x_1]$ in accordance with a linear law. Let us put

$$f(x) \approx f(x_0) + k(x - x_0),$$

where k is a number.

Since

$$f(x) = f(x_0) + f'(\xi)(x - x_0),$$

the error σ has an absolute value equal to

$$\sigma = |k - f'(\xi)| |x - x_0|.$$

Suppose we know that $|k - f'(x)| \leq M'$ for all x in $[x_0, x_1]$; since $|x - x_0| \leq |x_1 - x_0|$, we have

$$\varepsilon = M' |x_1 - x_0|.$$

This maximum error is suitable for all x in $[x_0, x_1]$.

Examples. (1) We have for the value of $\ln 783$ ($= 6.66312$) calculated above:

$$\left| \frac{1}{785} - \frac{1}{781 + \theta \cdot 2} \right| \cdot 2 = \frac{4 - \theta \cdot 2}{785(781 + \theta \cdot 2)} \cdot 2$$

i.e. the estimate holds:

$$\frac{4 - \theta \cdot 2}{785(781 + \theta \cdot 2)} \cdot 2 < \frac{8}{785 \cdot 781} < 0.000014 = \varepsilon.$$

If, for all x in the interval $[781, 789]$, we use the formula

$$\ln x = \ln 781 + \frac{1}{785} (x - 781)$$

as we did for $x = 782$ and $x = 783$, we get:

$$\sigma = \left| \frac{1}{785} - \frac{1}{781 + \theta(x - 781)} \right| (x - 781) = \frac{4 - \theta(x - 781)}{785[781 + \theta(x - 781)]} (x - 781);$$

the maximum error ε can be taken as 0.00006, since

$$\sigma = \frac{4}{785 \cdot 781} \cdot 8 = \frac{32}{785 \cdot 781} < 0.00006.$$

(2) The error of the approximation (Sec. 53)

$$\ln(x + h) \approx \ln x + \frac{h}{x}$$

has an absolute value equal to

$$\sigma = \left| \frac{1}{x} - \frac{1}{x + \theta h} \right| |h| = \frac{\theta h^2}{x(x + \theta h)} < \frac{h^2}{x^2},$$

i.e. we can take

$$\varepsilon = \frac{h^2}{x^2}.$$

This is an important complement to the method indicated for calculating logarithms. It is clear, for instance, from the expression for ε , that as x increases we can increase h proportionately, without exceeding the indicated maximum error.

(3) We can find similarly the maximum error, obtained in Sec. 53, for the value of $\sin 31^\circ = 0.5151$; we first write down the error:

$$\left[\cos \frac{\pi}{6} - \cos \left(\frac{\pi}{6} + \theta \cdot \frac{\pi}{180} \right) \right] \cdot \frac{\pi}{180} = 2 \sin \left(\frac{\pi}{6} + \theta \cdot \frac{\pi}{360} \right) \sin \theta \frac{\pi}{360} \cdot \frac{\pi}{180}$$

then, by using the inequalities

$$\sin \alpha < \alpha, \quad 0 < \alpha < \frac{\pi}{2},$$

arrive at the maximum error:

$$\varepsilon = 2 \left(\frac{\pi}{6} + \frac{\pi}{360} \right) \cdot \frac{\pi}{360} \cdot \frac{\pi}{180} \approx 0.0002.$$

67. Behaviour of a function in an interval. We can now undertake the investigation of the variation of a function in an interval (see Sec. 15, IV) with the aid of the derivative. It will be assumed, unless there is some special proviso, that the function always has a derivative.

DIRECT THEOREM. If a function $f(x)$ is increasing in an interval, its derivative $f'(x)$ is non-negative;

if $f(x)$ is decreasing in an interval, its derivative $f'(x)$ is non-positive;

if $f(x)$ does not vary (is constant) in an interval, its derivative $f'(x)$ vanishes identically.

Proof. If a function is increasing throughout an interval, it is increasing at every point of the interval; hence, by what was proved in Sec. 64, every numerical derivative of the function is non-negative; the derivative may similarly be seen to be non-positive when the function is decreasing; finally, the derivative of a constant vanishes.

Thus the derivative cannot change sign in an interval where the function is monotonic.

This enables us to establish the sign of the derivative from the nature of the variation of a function in a given interval. The converse is much more important, however, since it reduces the question of the variation of a function in an interval to the simpler question of the sign of another function—namely, the derivative—in this interval.

CONVERSE. If the derivative $f'(x)$ of a function $f(x)$ is positive everywhere in an interval, $f(x)$ is increasing in the interval;

if the derivative $f'(x)$ of a function $f(x)$ is everywhere negative in an interval, $f(x)$ is decreasing in the interval;

if the derivative $f'(x)$ of $f(x)$ vanishes everywhere in an interval, $f(x)$ does not vary (is constant) in the interval.

Proof. Let x_1 and x_2 be two arbitrary points in the interval, and let $x_1 < x_2$. We have by Lagrange's formula:

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1), \quad x_1 < \xi < x_2.$$

If the derivative is positive everywhere, $f'(\xi) > 0$, so that $f(x_2) > f(x_1)$ for any x_1 and x_2 , $x_1 < x_2$, i.e. the function is increasing; if the derivative is everywhere negative, $f'(\xi) < 0$, so that $f(x_2) < f(x_1)$ for any x_1 and x_2 , $x_1 < x_2$, i.e. the function is decreasing; finally if the derivative is everywhere zero, $f'(\xi) = 0$, so that $f(x_2) = f(x_1)$ for any x_1 and x_2 , i.e. the function is constant.

We see that *a function is monotonic in an interval in which its derivative has constant sign.*

This proposition gives us a simple and convenient analytic test for the monotonicity of a function in an interval.

For instance, to verify that the function $y = \frac{2}{3}x^3 - 2x^2 - 6x + 3$ is decreasing in the interval $-1 < x < 3$, we only need to show that its derivative $y' = 2x^2 - 4x - 6$ is negative for $-1 < x < 3$. This is in fact the case, since $y' = 2(x+1)(x-3)$, the factor $x+1$ being positive for all the x in question, and the factor $x-3$ negative.

This converse enables us to state the following important theorem.

FIRST SUFFICIENT TEST FOR AN EXTREMUM. A point x_0 is an extremal point of a function $f(x)$ if the derivative $f'(x)$ changes sign as x passes through x_0 ; a change of sign from $+$ to $-$ gives a maximum, and from $-$ to $+$ a minimum. At x_0 itself the derivative either vanishes or does not exist.

Proof. Let the derivative change sign from $+$ to $-$ as x passes through x_0 ; this means that there is an interval to the left of x_0 in which the function is increasing, and to the right an interval of decrease of the function. Hence x_0 is a maximum point of the function.

Similarly, if the sign of the derivative changes from $-$ to $+$, x_0 is a minimum point of the function.

REMARK 1. The change of sign of the derivative is not in general a necessary test for an extremal point; for instance, if $f(x)$ is "infinitely oscillating" at x_0 , we cannot speak of a change of sign of $f'(x)$ as x passes through x_0 , since $f(x)$ is not monotonic in any interval beginning or ending in the point x_0 ; in every such interval $f'(x)$ has values of different signs, and at the same time x_0 may be an extremal point of $f(x)$.

REMARK 2. It should also be borne in mind that we cannot conclude the existence of an extremum merely from a change of sign of the derivative; we also need to know that the function itself is continuous at the point. For instance, let $y = 1/x^2$. The derivative of this function $y' = -2/x^3$ changes sign as x passes through $x = 0$: to the left of zero $y' > 0$, i.e. the function is increasing, to the right of zero $y' < 0$, i.e. the function is decreasing; at $x = 0$ itself there is no maximum, whatever the value we assign to the function at $x = 0$, since the function has an infinite discontinuity at $x = 0$.

This situation is evident from the graph of the function (Fig. 78).

If the function has no derivative at certain points, we have to consider the behaviour of the function in the intervals between these points.

We can now sum up, and indicate the sequence of operations for investigating the variation and finding the extrema of a given continuous function $f(x)$ in a given interval $[a, b]$:

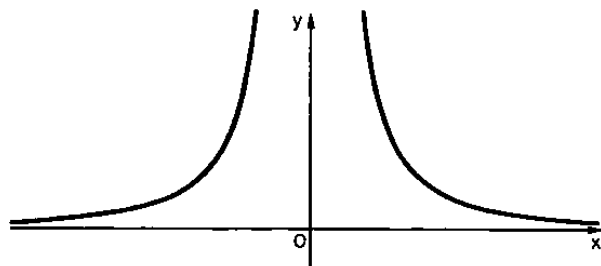


FIG. 78

(1) We must first of all find the points of the interval at which the derivative does not exist, and the stationary points of $f(x)$ (i.e. the real roots of the equation $f'(x) = 0$). There can only be a limited number of these points in an interval $[a, b]$ for the functions considered by us (i.e. excluding the “infinitely oscillating” type); let us write them in increasing order as $x_1, x_2, \dots, x_n$. These are the points of the interval where $f(x)$ may have extrema; they are sometimes called “critical points”.

(2) We next subdivide $[a, b]$ with the aid of points x_i into the sub-intervals $[a, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n], [x_n, b]$, in each of which the derivative retains a constant sign. For if this were not the case, the derivative would vanish (or not exist) at points additional to those of our subdivision*. These intervals are therefore intervals in which the function is monotonic.

(3) Further, the nature of the variation of the function in each of the intervals $[x_i, x_{i+1}]$ is established from the sign of the derivative,

* Strictly speaking, this assertion requires proof. Let the derivative not keep the same sign in an interval $[x_i, x_{i+1}]$. We have to show that, in this case, $f'(x)$ either vanishes or does not exist inside the interval $[x_i, x_{i+1}]$. In fact, since $f'(x)$ changes sign in $[x_i, x_{i+1}]$ (say from $+$ to $-$), $f(x)$ changes the nature of its variation (from increase to decrease), which is only possible if there is an extremal point inside $[x_i, x_{i+1}]$; but at an extremal point $f'(x)$ necessarily vanishes, provided it exists.

i.e. the nature of each of the points x_i . It may happen that some x_i are not extremal points of the function. This occurs if the function is monotonic in the same sense in two neighbouring intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ separated by the point x_i (the derivative has the same sign in them). They then combine into a single interval of monotonicity of the function. In this case x_i (if $f'(x_i) = 0$) remains a "stationary point" but is not an extremal point (e.g. $x = 0$ for the function $y = x^3$).

(4) Finally, the extremal values of $f(x)$ are calculated simply by substituting the values found for x —the "extremal points"—in the expression for the function.

As regards the greatest value (absolute maximum) of the function in the interval $[a, b]$, it will evidently be the greatest of the maxima in the interval taken in conjunction with the values of the function at the ends. Hence to find the greatest value of a function $f(x)$ in an interval $[a, b]$, we have to associate the end-values $f(a)$ and $f(b)$ with the maxima inside the interval, and choose the greatest from among these numbers. Instead of the maximum values we can simply take the values of the function at all the stationary points.

Finding the least (absolute minimum) value of a function in an interval is precisely similar.

The following general remarks may simplify the investigation of functions given by complicated expressions.

The extremal points and intervals of monotonicity do not change if the function is multiplied by a positive constant, and only change their sense if multiplied by a negative constant. The extremal points and intervals of monotonicity of a function of a function $f[\varphi(x)]$ are the same as the extremal points and intervals of monotonicity of function $\varphi(x)$, if $f(u)$ is monotonic, the nature of the extrema being the same if $f(u)$ is increasing and in the opposite sense if $f(u)$ is decreasing. The arithmetic sum of similarly monotonic functions is monotonic; the product of similarly monotonic functions, having a common sign everywhere, is monotonic, and so on.

These assertions follow directly from the above analytic tests.

For instance, to show that $\ln\left(e^x + \frac{x}{\ln x}\right)$ is increasing for $x > e$, we only need to show that $\frac{x}{\ln x}$ has this property; to investigate the extrema of $e^{\sqrt[3]{(x \sin x)^3}}$ it is sufficient to find the extrema of $t = (x \sin x)^2$, since $u = \sqrt[3]{t}$ and e^u are both increasing functions.

68. Examples.

I. BASIC ELEMENTARY FUNCTIONS. Although the behaviour of each of the elementary functions has already been described by us with the aid of their graphs in Chapter I, the differential calculus enables

some of their basic properties to be discovered very easily analytically.

Power functions: $y = x^n$. The derivative $y' = nx^{n-1}$ is positive for $x > 0$ if $n > 0$, and negative if $n < 0$. Hence the function is either always increasing on the positive semi-axis Ox ($n > 0$) or always decreasing ($n < 0$).

Exponential functions: $y = a^x$. The sign of the derivative $y' = \ln a \cdot a^x$ is the same as the sign of $\ln a$, since a^x is always positive. Hence $y' > 0$ if $a > 1$, and $y' < 0$ if $a < 1$. The exponential function is monotonic throughout Ox ; it is increasing in the first case, and decreasing in the second.

Logarithmic functions: $y = \ln x$. The derivative $y' = 1/x$ is positive for $x > 0$ (and the function is only defined for these values), i.e. $\ln x$ is always increasing.

The course of variation of the other basic elementary functions may similarly be established by considering the derivative.

II. THE ELEMENTARY FUNCTIONS.

(1) *Quadratic forms.* Let us investigate the familiar function

$$y = ax^2 + bx + c.$$

On equating its derivative to zero: $y' = 2ax + b = 0$, we get $x = -b/2a$. The quadratic form thus has only one extremal point $x = -b/2a$, and two intervals of monotonicity $(-\infty, -b/2a)$, $(-b/2a, \infty)$.

The sort of extremum obtained with this value of the independent variable depends on the sign of a . For we can write the derivative as $y' = 2a(x + b/2a)$, whence it is clear that, if a is positive, the derivative is negative for $x < -b/2a$ and positive for $x > -b/2a$ whilst it has the opposite signs in these intervals if a is negative.

Thus with $a > 0$ the point $x = -b/2a$ represents a minimum, and the corresponding value of the function

$$y = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c = -\frac{b^2 - 4ac}{4a}$$

is its minimum value, this being at the same time its least value in any interval containing the point $x = -b/2a$.

With $a < 0$, $x = -b/2a$ will be a maximum point, and the corresponding value of the function $y = -(b^2 - 4ac)/4a$ is its maximum value, this being at the same time the greatest value in any interval containing the point $x = -b/2a$.

In either case, the point with co-ordinates $x = -b/2a$, $y = -(b^2 - 4ac)/4a$ is the vertex of the parabola $y = ax^2 + bx + c$.

Of course all this is in complete agreement with what was said in Sec. 18.

(2) Cubic functions. Let us consider

$$y = 3x^3 + 4.5x^2 - 4x + 1.$$

We equate the derivative to zero:

$$9x^2 + 9x - 4 = (3x - 1)(3x + 4) = 0.$$

Clearly, the derivative vanishes for $x = -4/3$ and $x = 1/3$.

Since both factors are negative for $x < -4/3$, their product is positive for these values of x , i.e. the function is increasing. With $-4/3 < x < 1/3$, the derivative is negative and the function decreasing, whilst with $x > 1/3$ the derivative is positive again and the function increasing. Hence $x = -4/3$ is a maximum point, and $x = 1/3$ a minimum point, whilst we have three intervals of monotonicity: from $-\infty$ to $-4/3$ the function is increasing, from $-4/3$ to $1/3$ decreasing, and from $1/3$ to $+\infty$ increasing.

We can deduce from this analysis that the equation

$$3x^3 + 4.5x^2 - 4x + 1 = 0$$

has only one real (negative) root, i.e. its other two roots* are complex conjugates.

For, with $x = -4/3$ our function takes the positive value $y = 7\frac{2}{9}$; since the function is increasing for $x < -4/3$ and becomes negative for sufficiently large negative x^{**} , its graph cuts the axis of abscissae at least once in the interval from $-\infty$ to $-4/3$; on the other hand, from $-4/3$ to $+\infty$ the function takes its least value at $x = 1/3$, and this value is positive: $y = 5/18$, so that the graph never cuts Ox in the interval $(-4/3, +\infty)$.

(3) Damped oscillations. Let us consider the function $y = e^{-x} \sin x$ for $x > 0$. Its derivative $y' = e^{-x}(\cos x - \sin x) = \sqrt{2}e^{-x} \sin\left(\frac{\pi}{4} - x\right)$ vanishes at the points $\pi/4, 5\pi/4, 9\pi/4, \dots$, which divide the positive semi-axis Ox into intervals in which the derivative has alternate signs. These points therefore give in turn maxima and minima of the function.

Since $-1 \leq \sin x \leq 1$, whilst $e^{-x} > 0$, we have

$$-e^{-x} \leq y \leq e^{-x}$$

and the graph lies wholly inside the domain bounded by the curves $y = e^{-x}$ and $y = -e^{-x}$ (Fig. 79). As $x \rightarrow \infty$ the factor e^{-x} tends to zero, whilst the second

* The number of roots of any algebraic equation is equal to its degree.

** This follows from the fact that

$$x = 3x^3 + 4.5x^2 - 4x + 1 = x^3 \left(3 + \frac{4.5}{x} - \frac{4}{x^2} + \frac{1}{x^3} \right)$$

has the same sign as x^3 for sufficiently large $|x|$, since the expression in brackets becomes as near 3 as desired.

factor $\sin x$ remains bounded, so that $e^{-x} \sin x$ also tends to zero. As x increases the derivative also tends to zero; consequently, on moving away from the origin, the point of the curve approaches Ox by a more and more gently sloping path whilst oscillating round it. The graph of the function (we recommend the reader to verify this!) touches the curves $y = -e^{-x}$ at the x for which $\sin x = +1$ (i.e. at $x = \pi/2, 3\pi/2, 5\pi/2, \dots$); it is noteworthy that these points are not the extremal points of the function (which are $\pi/4, 5\pi/4, 9\pi/4, \dots$).

The function $y = e^{-x} \sin x$ defines the law of a so-called damped vibration. The factor e^{-x} not only "damps down" the swings of the vibration originating from the second periodic factor $\sin x$, but also displaces the vertices.

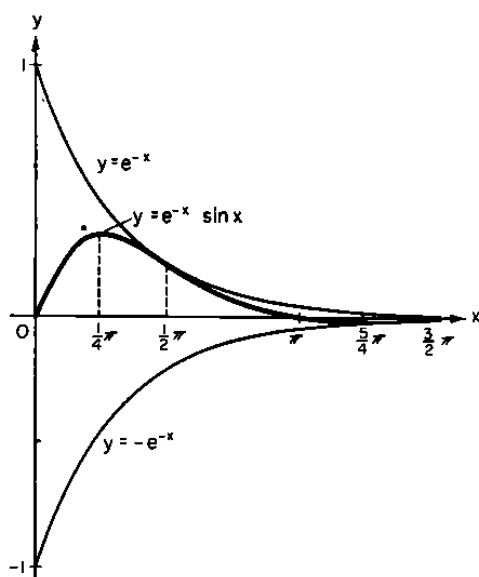


FIG. 79

III. INEQUALITIES. The methods for investigating functions can often be used for proving inequalities.

Let us show, for instance, that

$$\frac{2}{\pi} x < \sin x < x, \quad \text{if } 0 < x < \frac{\pi}{2}.$$

We take the function $y = \frac{\sin x}{x}$. Since its derivative

$$y' = \frac{\cos x}{x^2} (x - \tan x)$$

is negative in the interval $(0, \frac{1}{2}\pi)$ ($x < \tan x$), y is decreasing, so that

$$\frac{\sin(\pi/2)}{\pi/2} < \frac{\sin x}{x} < 1,$$

whence our inequalities follow.

The inequality

$$\sin x > x - \frac{x^3}{6}$$

holds for any $x > 0$.

To prove this we take the function

$$f(x) = \sin x - x + \frac{x^3}{6}$$

and show that it is always positive if $x > 0$. Since $f(0) = 0$, it is sufficient to show that $f(x)$ is increasing in the interval $(0, \infty)$. The derivative

$$f'(x) = \cos x - 1 + \frac{x^2}{2}$$

must be positive, though this is not directly obvious. However, its derivative

$$f''(x) = x - \sin x$$

is positive for $x > 0$, so that $f'(x)$ is an increasing function; on noticing that $f'(0) = 0$, we conclude that $f'(x) > 0$. The increase of $f(x)$ follows in turn from this.

The inequality

$$x - \sin x < \frac{x^3}{6}$$

enables us to find the maximum error in replacing $\sin x$ by x .

Let us prove the inequality

$$x > \ln(1 + x)$$

for any $x > 0$. The function $f(x) = x - \ln(1 + x)$ vanishes for $x = 0$: $f(0) = 0$, whilst it is increasing for $x > 0$, as is easily seen from the derivative: $f'(x) = 1 - 1/(1 + x)$, which is positive for all $x > 0$. Consequently, $f(x) > 0$ in the interval $(0, \infty)$, whence the required inequality follows.

IV. PROBLEMS ON GREATEST AND LEAST VALUES. Suppose we are given two magnitudes, connected by a functional relationship, and that we want to find the value of one of them (lying in some interval*), for which the other takes its greatest or least possible value.

* Which may in fact be unbounded.

To solve such a problem, we must first of all form an analytic expression for the function with the aid of which one quantity is expressed in terms of the other; then we must find the greatest or least value of the function in the given interval.

Examples. (1) Let us find the least length of intercept between the co-ordinate axes that passes through a given point (x_0, y_0) . We can obviously assume that (x_0, y_0) lies in the first quadrant. On

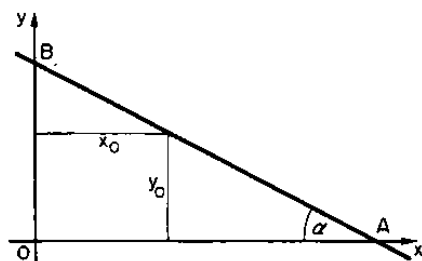


FIG. 80

choosing the angle α (Fig. 80) as the independent variable, and writing l for the length of intercept AB , we get:

$$l = \frac{x_0}{\cos \alpha} + \frac{y_0}{\sin \alpha}$$

where α varies in the interval $(0, \frac{1}{2}\pi)$. Hence

$$\frac{dl}{d\alpha} = \frac{x_0 \sin \alpha}{\cos^2 \alpha} - \frac{y_0 \cos \alpha}{\sin^2 \alpha} = \frac{y_0 \sin \alpha}{\cos^2 \alpha} \left(\frac{x_0}{y_0} - \cot^3 \alpha \right).$$

Since the first factor is positive, the sign of the derivative is the same as the sign of the second factor. Bearing in mind that $\cot \alpha$ decreases continuously from $+\infty$ to 0 in the interval $(0, \frac{1}{2}\pi)$, we conclude that, as α varies from 0 to $\frac{1}{2}\pi$, the derivative is first of all negative, vanishes for $\alpha = \alpha_0$ where $\cot \alpha_0 = \sqrt[3]{x_0/y_0}$, then is positive. Thus our function has one minimum in the given interval, at the point $\alpha = \alpha_0$, and this is the required least value:

$$l_{\min} = \frac{x_0}{\cos \alpha_0} + \frac{y_0}{\sin \alpha_0}.$$

But $\cot \alpha_0 = \sqrt[3]{x_0/y_0}$, so that we have, on expressing $\cos \alpha_0$ and $\sin \alpha_0$ in terms of $\cot \alpha_0$:

$$l_{\min} = \frac{x_0}{\sqrt[3]{x_0/y_0}} \sqrt{1 + \sqrt[3]{\left(\frac{x_0}{y_0}\right)^2}} + y_0 \sqrt{1 + \sqrt[3]{\left(\frac{x_0}{y_0}\right)^2}} = (x_0^{\frac{2}{3}} + y_0^{\frac{2}{3}})^{\frac{3}{2}}.$$

When solving concrete problems on greatest and least values, the choice of independent variable is most often left to us, and the ease and speed of obtaining the result depends on our choice being a good one.

In the problem above, for instance, we might have taken $OA = x$ as the independent variable. Then

$$l = \frac{x}{x - x_0} \sqrt{(x - x_0)^2 + y_0^2}$$

and this expression is less convenient to investigate than the above.

(2) Let us find the greatest among the numbers $1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \dots, \sqrt[n]{n}, \dots$. We shall regard these numbers as the values at "integral points" $x = 1, 2, 3, 4, \dots, n, \dots$ of a function $y = \sqrt[x]{x}$, continuous for $x > 0$. Let us investigate $y = x^{1/x}$. We have:

$$y' = x^{\frac{1}{x}-2}(1 - \ln x).$$

The expression for the derivative enables us to conclude that the function is increasing in the interval $(0, e)$, is decreasing in (e, ∞) , and has a single maximum, equal to $e^{1/e}$, at $x = e$.

Hence the required number can only be one of the two values of $y = x^{1/x}$ corresponding to the integral values of x nearest to the maximum point $x = e$. These values are $\sqrt{2}$ and $\sqrt[3]{3}$. But since $\sqrt[3]{3} > \sqrt{2}$ (the proof of which is elementary), the greatest among the numbers $1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \dots, \sqrt[n]{n}, \dots$ is $\sqrt[3]{3}$.

69. A property of the primitive. We shall now consider, as an example of an important application of the theorem on the behaviour of a function in an interval (Sec. 67), a proposition which plays an essential role in constructing the integral calculus (Chapter V and VI). This proposition is concerned with a new concept which is the inverse of the derivative, namely, that of *primitive of a function*.

Definition. The *primitive of a function* $f(x)$ is the function $F(x)$, the derivative of which is equal to the given function:

$$F'(x) = f(x).$$

In order to gain a better idea of the relationship between a primitive of a function and the function itself we shall consider some examples.

Let $y = x^2$. For what function is x^2 a primitive? Obviously, for $x^3/3$:

$$\left(\frac{x^3}{3}\right)' = x^2.$$

A primitive of x^2 is therefore $x^3/3$. But this is not the only one.

The derivative of $x^3/3 + 5$, of $x^3/3 - 100$, and in general of $x^3/3 + C$, where C is an arbitrary constant, is also x^2 . Thus any function $x^3/3 + C$ is a primitive of x^2 .

The two functions x^2 and $x^3/3 + C$ are thus related to each other: *the first is the derivative of the second, the second is the primitive of the first.*

Similarly, any function $\frac{x^{k+1}}{k+1} + C$ is the primitive of $x^k (k \neq -1)$; any function $\sin x + C$ is a primitive of $\cos x$; any function $\ln x + C$ is a primitive of $1/x$ and so on.

THEOREM. If a function has a primitive, it has an infinite set of them, any two of which differ only by a constant.

Proof. Let $F(x)$ be a primitive of the function $f(x)$, i.e. $F'(x) = f(x)$; then $F(x) + C$, where C is an arbitrary constant, is also a primitive, since

$$[F(x) + C]' = F'(x) = f(x).$$

It remains for us to show that any two primitives of $f(x)$ differ only by a constant. Let $F(x)$ and $\Phi(x)$ be two primitives of $f(x)$ which are not identically equal. We have:

$$F'(x) = f(x), \Phi'(x) = f(x).$$

Subtraction of one equation from the other gives: $F'(x) - \Phi'(x) = 0$, i.e. $[\Phi(x) - F(x)]' = 0$. But if the derivative of a function (in our case $\Phi(x) - F(x)$) vanishes identically, the function is constant (Sec. 67); hence $\Phi(x) - F(x) = C_1$, where C_1 is a definite constant.

This is what we wanted to prove.

Thus the formula $F(x) + C$, where $F(x)$ is a primitive of $f(x)$ and C is an arbitrary constant, embraces all the primitives of $f(x)$. On assigning different numerical values to C , we get different primitives; and there is no primitive that cannot be obtained from this formula by assigning a suitable value to C .

Using this last theorem, simple relationships can be obtained between functions whose derivatives are equal.

For example, let us take the two functions: $\arctan x$ and $\arctan \frac{1+x}{1-x}$. Their derivatives are equal:

$$\arctan \frac{1+x}{1-x} = \arctan x + C, \quad -1 < x < 1;$$

on setting $x = 0$ here, we find that $C = \arctan 1$, i.e.

$$\arctan \frac{1+x}{1-x} = \arctan x + \arctan 1 = \arctan x + \frac{\pi}{4}.$$

This can also be proved by elementary means.

Similarly,

$$\arcsin x = \arctan \frac{x}{\sqrt{1-x^2}}, \quad -1 < x < 1.$$

For,

$$(\arcsin x)' = \left(\arctan \frac{x}{\sqrt{1-x^2}} \right)' = \frac{1}{\sqrt{1-x^2}}.$$

Therefore

$$\arcsin x = \arctan \frac{x}{\sqrt{1-x^2}} + C.$$

On putting $x = 0$, we find that $C = 0$. This formula is likewise easily proved by elementary means.

3. Applications of the Second Derivative

70. Second sufficient test for an extremum. Since the second derivative $f''(x)$ of a function $f(x)$ is the derivative of the derivative $f'(x)$, the discussion of the previous article enables us to describe with the aid of $f''(x)$ the behaviour of $f'(x)$, which in turn permits a sounder judgement of the variation of $f(x)$ itself.

We first of all apply the second derivative to finding the extrema of a function.

SECOND SUFFICIENT TEST FOR AN EXTREMUM. If $f'(x_0)$ vanishes, whilst $f''(x_0)$ does not vanish, x_0 is an extremal point of $f(x)$; it is a maximum point if $f''(x_0) < 0$ and a minimum point if $f''(x_0) > 0$.

Proof. Since $f'(x_0) \neq 0$, $f'(x)$ is either increasing (if $f''(x_0) > 0$) or decreasing (if $f''(x_0) < 0$) at the point x_0 . This means, in view of the condition $f'(x_0) = 0$, that $f'(x)$ has different signs to the left and right of x_0 in some neighbourhood of it: $-$ to the left and $+$ to the right in the first case ($f'' > 0$) and $+$ to the left and $-$ to the right in the second case ($f'' < 0$). For, if a function ($f'(x)$ here) vanishes at a point and is increasing, it must have values less than zero to the left of the point, i.e. negative values, and greater i.e. positive values to the right. Similarly, if $f'(x)$ vanishes at a point and is decreasing, it has positive values to the left of the point and negative to the right.

Thus $f'(x)$ changes sign as x passes through x_0 , from $-$ to $+$ if $f''(x_0) > 0$, and from $+$ to $-$ if $f''(x_0) < 0$, and this implies, by our first test (Sec. 67), that $f(x)$ has an extremum at x_0 : a minimum in the first case ($f'' > 0$), a maximum in the second ($f'' < 0$).

If $f'(x_0) = 0$ and $f''(x_0) = 0$, the second test cannot be applied and we have to go back to the first, "stronger" test.

For instance, the first two derivatives of $y = x^4$ both vanish at $x = 0$; yet the function has a minimum at this point, as indicated by the first test: the derivative $y' = 4x^3$ changes sign from $-$ to $+$ as x passes through zero. The function $y = x^3$ has no extremum at $x = 0$ although its first two derivatives both vanish there; and in fact, its first derivative does not change sign as x passes through zero.

The second test is very convenient, however, when applicable; instead of considering the sign of $f'(x)$ at points distinct from the suspected extremal point, we can get an answer from the sign of $f''(x)$ at this same point.

Examples. (1) $f(x) = ax^2 + bx + c$. Since $f'(x) = 2ax + b$, it vanishes for $x = -b/2a$, whilst $f''(x) = 2a$; thus $f(x)$ has a maximum at this point if $a < 0$, and a minimum if $a > 0$.

This example should give the reader a clear picture of the great value of theory for practical mathematics. It shows that, with an extension of the methods of analysis, the arguments and working needed for investigating concrete problems are simplified and shortened. In Chapter I, where we had no command of the differential calculus, the quadratic function required fairly lengthy and special discussion from us; in the first article of the present chapter, where use was made only of the first derivative, we obtained the same results much more simply; finally, here we need only three lines, thanks to the further concept of second derivative.

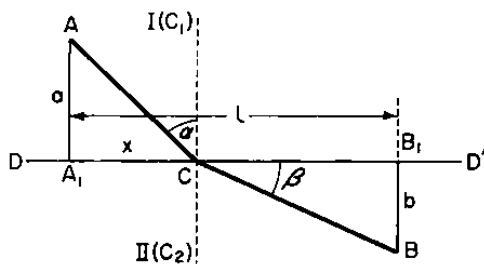


FIG. 81

(2) Let us find the path of a ray of light as it passes from one medium to another.

Let the ray start from point A of the first medium, where the velocity of light is c_1 , and leave at a point B of the second medium, in which the velocity is c_2 , the two media being separated by a plane (Fig. 81). The required path consists of two straight pieces AC and CB , lying in the same plane as the perpendicular to the boundary plane at the point C . Our problem obviously

consists in finding the position of point C . The solution starts from Fermat's important principle: the trajectory of a light ray is the curve such that light propagated along it covers a distance in the shortest time. In our case the time taken for the light to pass from point A to point C is equal (see the notation in the figure) to $\frac{\sqrt{a^2 + x^2}}{c_1}$, whilst the time of passing from point C to point B is equal to $\frac{\sqrt{b^2 + (l-x)^2}}{c_2}$. Thus the light moves from the initial to the final point in time

$$t = \frac{1}{c_1} \sqrt{a^2 + x^2} + \frac{1}{c_2} \sqrt{b^2 + (l-x)^2}.$$

By Fermat's principle, the problem amounts to finding the least value of this function (in $-\infty < x < \infty$). We have:

$$\frac{dt}{dx} = \frac{1}{c_1} \cdot \frac{x}{\sqrt{a^2 + x^2}} - \frac{1}{c_2} \cdot \frac{l-x}{\sqrt{b^2 + (l-x)^2}}.$$

Since $(dt/dx)_{x=0} < 0$, whilst $(dt/dx)_{x=l} > 0$, the derivative vanishes inside the interval $(0, l)$. But the second derivative

$$\frac{d^2t}{dx^2} = \frac{1}{c_1} \frac{a^2}{(\sqrt{a^2 + x^2})^3} + \frac{1}{c_2} \frac{b^2}{(\sqrt{b^2 + (l-x)^2})^3}$$

is positive for all x , $d^2t/dx^2 > 0$, i.e. dt/dx is always increasing, so that it cannot have more than one zero, which must be the required minimum point (because $d^2t/dx^2 > 0$); at this point the function has its least value. We have for the x at which dt/dx vanishes:

$$\frac{1}{c_1} \frac{x}{\sqrt{a^2 + x^2}} = \frac{1}{c_2} \frac{l-x}{\sqrt{b^2 + (l-x)^2}},$$

i.e. (see Fig. 81):

$$\frac{1}{c_1} \frac{A_1C}{AC} = \frac{1}{c_2} \frac{B_1C}{BC}$$

or

$$\frac{1}{c_1} \sin \alpha = \frac{1}{c_2} \sin \beta,$$

whence

$$\frac{\sin \alpha}{\sin \beta} = \frac{c_1}{c_2}.$$

Therefore, the ratio of the sine of the angle of incidence to the sine of the angle of refraction is equal to the ratio of the velocities of propagation of the light. This is the familiar law of refraction of light.

71. Convexity and concavity of a curve. Points of inflexion.

I. *Definition.* An arc is said to be **convex** (Sec. 57) if it cuts any chord at only two points.

Arc AB in Fig. 82 is convex, whilst the arc in Fig. 83 is not convex: there are chords M_1M_2 in Fig. 83 which cut arc AB in points M_3 additional to M_1 and M_2 .

We shall only consider curves in the Cartesian co-ordinate plane which are the graphs of single-valued and continuous functions $y = f(x)$. If such a curve is convex, its convexity is seen either from above (from the side of positive ordinates) or from below (from the side of negative ordinates). Strictly speaking, the curve $y = f(x)$ is convex upwards if an arbitrary arc of it lies above the chord subtending the arc; it is convex downwards if any arc of it lies below the chord subtending the arc.

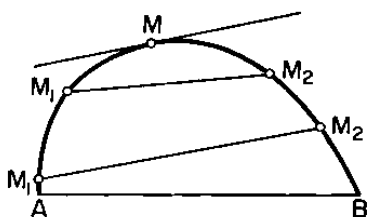


FIG. 82

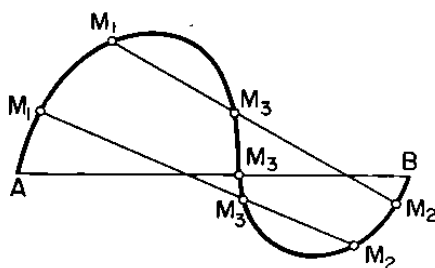


FIG. 83

It has become customary to simply call the curve convex in the first case, and concave in the second. In Fig. 84 the arc AC is convex, and arc CB concave. It is quite clear that a convex arc lies below, and the concave arc above any tangent to it. The converse is also obvious: if a curve has a tangent at every point and lies below it, the curve is convex, and if the curve lies above any tangent, it is concave.

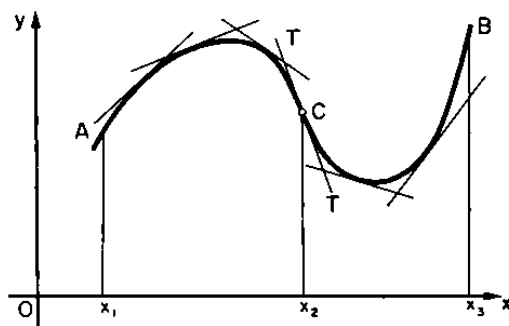


FIG. 84

We take curve AB (Fig. 84), which is neither convex nor concave. The point C on the curve, at which the "break" occurs in the convexity, is called a point of inflexion.

Definition. A point of inflexion is a point separating a convex from a concave part of a curve.

If the curve has a tangent at every point, i.e. the function $f(x)$ has a derivative (finite or infinite), the tangent at a point of inflexion cuts the curve in such a way that the curve lies on both sides of the tangent in any neighbourhood of the point.

II. We now turn to the connection between the nature of the convexity of a curve $y = f(x)$ and the properties of the function $f(x) \cdot (f(x))$ is assumed to be at least twice differentiable.)

We take the extremal points of $f'(x)$. These points (if they exist) separate the entire interval in which the function is considered into intervals in which $f'(x)$ is monotonic, i.e. in which $f''(x)$ has constant sign. The graph of $f(x)$ in an interval in which $f''(x)$ has constant sign is convex from one side or the other.

DIRECT THEOREM. If a curve $y = f(x)$ is convex, $f''(x)$ is non-positive in the corresponding interval; if $y = f(x)$ is a concave curve, $f''(x)$ is non-negative in the corresponding interval.

For, if the curve is convex, any tangent lies above it, and as the point of contact moves from left to right, the angle formed by the tangent with Ox can only diminish, this being evident geometrically (see Fig. 84) (we consider here, instead of an obtuse angle, the complementary angle taken with the $-$ sign). As the angle diminishes, so does its tangent, i.e. $f'(x)$; hence the interval in question is one in which $f'(x)$ is decreasing, i.e. its derivative $f''(x)$ cannot be positive in this interval.

We find similarly that, if the curve is concave, $f''(x) \geq 0$. For instance, the parabola $y = x^4$ is concave throughout Ox and $d^2y/dx^2 = 12x^2$ is always positive, except at $x = 0$, where it vanishes.

CONVERSE THEOREM. If $f''(x)$ is negative everywhere in an interval, the curve $y = f(x)$ is convex in the interval.

If $f''(x)$ is everywhere positive in an interval, the curve $y = f(x)$ is concave in the interval.

Let $f''(x) < 0$. This implies that $f'(x)$ is decreasing, i.e. the slope of the tangent is decreasing, and similarly for the angle formed by the tangent with Ox . The curve obviously lies below every tangent in this case, i.e. it is convex. Similarly, it can be seen geometrically that the curve is concave if $f''(x) > 0$.

The analytic proof of the theorem is based on the observation that, if an arc is convex, the increment of the ordinate is less than the increment of the

ordinate of the corresponding tangent (Fig. 85); conversely, if this is true for any point x_0 of the interval $[x_1, x_2]$ and for any increment h of the abscissa, $x_1 \leq x_0 + h \leq x_2$, then the arc AB lies below any tangent, i.e. it is convex.

We take the difference of the increments of the ordinates of the curve and tangent. It can be written by Lagrange's formula as

$$\Delta y - dy = [f(x_0 + h) - f(x_0)] - f'(x_0)h = [f'(x_0 + \theta h) - f'(x_0)]h, \quad 0 < \theta < 1.$$

We obtain by again applying Lagrange's formula:

$$\Delta y - dy = f''(x_0 + \theta_1 h)\theta h^2, \quad 0 < \theta < 1.$$

If $f''(x)$ is everywhere negative in the interval $[x_1, x_2]$, then $f''(x_0 + \theta_1 h) < 0$ and hence $\Delta y < dy$ for any $x_0, x_1 < x_0 < x_2$, and any $h, x_1 < x_0 + h < x_2$; the arc lies below the tangent, i.e. it is convex. The argument is similar when the sign of $f''(x)$ is positive.

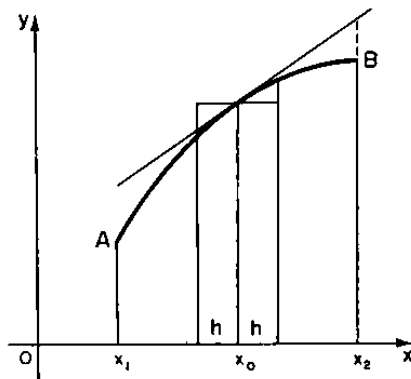


FIG. 85

III. The abscissa of a point of inflexion of a curve $y = f(x)$ separates intervals in which the first derivative $f'(x)$ is monotonic in opposite senses, and is therefore an extremum of $f'(x)$; conversely, if a point on Ox divides intervals in which $f'(x)$ is monotonic, being thus an extremal point of $f'(x)$, it is the abscissa of a point of inflexion of the curve $y = f(x)$. In view of this, finding the abscissae of points of inflexion and the intervals of convexity and concavity of the graph of $f(x)$ can be carried out by familiar rules.

We note first of all that the abscissa of a point of inflexion of the curve $y = f(x)$ is a real root of the equation $f''(x) = 0$, i.e. these abscissae are "stationary points" of $f'(x)$. We shall also give two sufficient tests for points of inflexion as extremal points of $f'(x)$.

FIRST TEST. If $f''(x_0) = 0$ and $f''(x)$ changes sign as x passes through x_0 , $(x_0, f(x_0))$ is a point of inflexion of the curve $y = f(x)$; a convex section lies to the left, and a concave to the right when the

sign changes from $-$ to $+$ and conversely, a concave section lies to the left and a convex to the right if the sign changes from $+$ to $-$; if $f''(x)$ does not change sign, the point x_0 is not the abscissa of a point of inflexion.

SECOND TEST. If $f''(x_0) = 0$, and $f'''(x_0) \neq 0$, the point $(x_0, f(x_0))$ is a point of inflexion of the curve $y = f(x)$; when $f'''(x_0) > 0$ a convex section lies to the left and a concave to the right, whilst if $f'''(x_0) < 0$ a concave section lies to the left and a convex to the right.

Examples. (1) The curve $y = x^5 + x$ has a point of inflexion at $(0, 0)$; to start with, $y''|_{x=0} = 20x^3|_{x=0} = 0$; then, since $y'''|_{x=0} = 0$, we see by considering the values of y'' close to $x = 0$ that y'' changes sign from $-$ to $+$ as x passes through $x = 0$; thus $x = 0$ is the abscissa of a point of inflexion, the curve having a convex portion to the left of this point and a concave to the right.

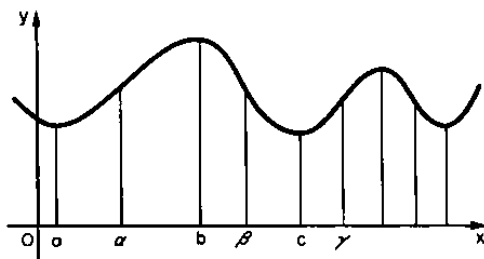


FIG. 86

(2) The curve $y = x^4 + x$ has no point of inflexion at $(0, 0)$; here also,

$$y''|_{x=0} = 12x^2|_{x=0} = 0$$

and

$$y'''|_{x=0} = 0,$$

but y'' does not change sign as x passes through $x = 0$.

We have assumed that $f(x)$ is twice differentiable throughout the interval in question. If this is not the case, we have to investigate $f'(x)$ and $f''(x)$ in the neighbourhoods of the individual points at which they do not exist.

In normal cases the extremal points $a, b, c, \dots$ and the abscissae of the points of inflexion $\alpha, \beta, \gamma, \dots$ alternate (Fig. 86).

A difference should be noted in our terms: an extremal point is a point on the axis of the independent variable, whilst a point of

inflexion is a definite point on the graph itself. This is bound up with the fact that the concept of extremal is relative, and dependent on the chosen co-ordinate system: a point on a curve corresponding to an extremal point in one system of co-ordinates may not correspond to an extremal point in another co-ordinate system; whereas the concept of point of inflexion does not depend on the co-ordinate system but is due to properties inherent in the curve itself: a point of inflexion remains so for any displacements of the curve on the plane (like a rigid body), i.e. whatever the co-ordinate system.

72. Examples.

I. BASIC ELEMENTARY FUNCTIONS. We are now in a position to see very easily the nature of the graphs of the basic elementary functions as regards convexity.

Thus the graph of the power function $y = x^n$ on the positive semi-axis Ox is concave for $n < 0$ and $n > 1$, and convex for $0 < n < 1$. For, $y'' = n(n-1)x^{n-2}$ is positive in the first cases and negative in the second.

The graph of the exponential function $y = a^x$ is concave throughout Ox for any $a > 0$.

The logarithmic $y = \log_a x$ is convex everywhere for $a > 1$ and concave for $a < 1$.

For,

$$(\log_a x)'' = -\frac{1}{\ln a} \frac{1}{x^2}$$

is negative in the first case and positive in the second.

Since

$$(\sin x)'' = -\sin x,$$

the intervals of convexity and concavity of the sine wave coincide respectively with the intervals of positive and negative sign of $\sin x$, whilst its points of inflexion are its points of intersection with Ox .

Starting from the convexity of the sine wave in the interval $(0, \frac{1}{2}\pi)$ and of the curve $y = \ln(1+x)$ in the interval $(0, \infty)$, the reader will easily give a further, geometrical proof of the inequalities (see Sec. 68, III):

$$\frac{2}{\pi} < \sin x < x \quad \text{for} \quad 0 < x < \frac{\pi}{2} \quad \text{and} \quad x > \ln(1+x) \quad \text{for} \quad x > 0.$$

II. VAN DER WAALS' EQUATION. It is not always possible to find the characteristic points of a function (its points of intersection with Ox , extremal points, abscissae of points of inflexion) by direct solution of the equations $f(x) = 0$, $f'(x) = 0$, $f''(x) = 0$. We must often have recourse to indirect considerations, following from the behaviour of $f(x)$, $f'(x)$, $f''(x)$ etc.

We shall illustrate what has been said by investigating the equation of state of a gas, due to Van der Waals:

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT,$$

where p is the pressure, v the volume, T the absolute temperature, a , b positive constants which differ for different gases, and R a constant, dependent on the number of gram-molecules but not on the composition of the gas.

The volume v must be greater than b ($v > b$), since with $v < b$ it would follow from the equation that the positive quantity RT was equal to a negative. By virtue of physical considerations, the values of v considered are large compared with b .

We shall consider the equation with various constant temperatures (isothermal processes). If $T = \text{const}$, p is a function of the single variable v :

$$p = \frac{RT}{v - b} - \frac{a}{v^2} = \varphi(v).$$

We take the first derivative of p with respect to v :

$$\frac{dp}{dv} = -\frac{RT}{(v - b)^2} + \frac{2a}{v^3} = \frac{1}{(v - b)^2} \left[2a \frac{(v - b)^2}{v^3} - RT \right].$$

To find directly the extremal points here implies solving a cubic equation. We get round this by observing that the sign of the derivative depends only on the second factor

$$\sigma = 2a \frac{(v - b)^2}{v^3} - RT,$$

which it is sufficient to investigate.

If $v = b$, $\sigma = -RT$, and as $v \rightarrow \infty$, $\sigma \rightarrow -RT$; since the derivative of σ with respect to v :

$$\frac{d\sigma}{dv} = 2a \frac{(v - b)(3b - v)}{v^4}$$

is positive in the interval $(b, 3b)$ and negative in the interval $(3b, \infty)$, i.e. σ has a maximum equal to $\frac{8a}{27b} - RT$ at the point $v_0 = 3b$, the question of the sign of σ is solved by the sign of $\frac{8a}{27b} - RT$.

Let it be negative, i.e. $T > \frac{1}{R} \frac{8a}{27b}$; then σ is negative for all v , $b < v < \infty$, and $p = \varphi(v)$ is always decreasing.

Further, let $\frac{8a}{27b} - RT > 0$, i.e. $T < \frac{1}{R} \frac{8a}{27b}$; then σ vanishes twice: at a point $v = v_1$, lying between b and $3b$, and at a point $v = v_2$, to the right of $3b$. It is clear that v_1 gives a minimum and v_2 a maximum.

Finally, let $\frac{8a}{27b} - RT = 0$, i.e. $T = \frac{1}{R} \frac{8a}{27b}$. Then σ vanishes only with $v = v_0 = 3b$, remaining negative for all other values of v . Hence dp/dv is also negative throughout the interval (b, ∞) , except for the point $v_0 = 3b$, where it has a maximum, equal to zero. Thus p is always decreasing, and the graph of $p = \varphi(v)$ has a point of inflexion at $v = v_0$ ($v_0 = 3b$, $p_0 = \frac{a}{27b^2}$), where the tangent is parallel to the Ov axis. The point v_0 is a stationary point of the function $\varphi(v)$. The values

$$T = T_0 = \frac{1}{R} \frac{8a}{27b}, \quad v = v_0 = 3b, \quad p = p_0 = \frac{a}{27b^2}$$

are known respectively as the critical temperature, volume and pressure of the gas.

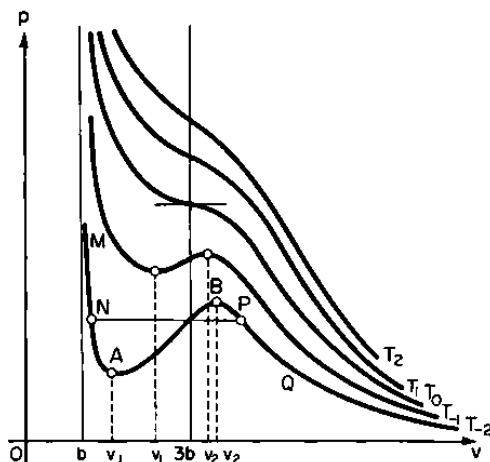


FIG. 87

Therefore, at temperatures less than critical ($\dots < T_{-2} < T_{-1} < T_0$), $p = \varphi(v)$ always has one minimum and one maximum point, which approach each other as the temperature rises; they coincide at $T = T_0$ with the point $v_0 = 3b$, which corresponds to a point of inflexion of the graph; at temperatures T above critical ($T_0 < T_1 < T_2 < \dots$), as also at $T = T_0$, $p = \varphi(v)$ is always decreasing. On noticing also that $p \rightarrow \infty$ as $v \rightarrow b$ and $p \rightarrow 0$ as $v \rightarrow \infty$, we can easily trace the graphs of functions $p = \varphi(v)$ corresponding to different temperatures. It is clear from Fig. 87 that the law of variation of the pressure p is only close to the Boyle-Mariotte law at fairly high (above critical) constant temperatures and large values of v . Otherwise the Boyle-Mariotte law is entirely inapplicable.

4. Auxiliary Problems. Solution of Equations

73. Cauchy's theorem and L'Hôpital's rule.

I. Lagrange's theorem proved in Sec. 65 is a particular case of the following.

CAUCHY'S THEOREM*. If functions $f(x)$ and $\varphi(x)$ are continuous in a closed interval $[x_1, x_2]$ and differentiable in the open interval (x_1, x_2) , whilst $f'(x)$ and $\varphi'(x)$ have no common zero** in the interval and $\varphi(x_1) \neq \varphi(x_2)$, there exists at least one $x = \xi$ in the interval for which

$$\frac{f(x_2) - f(x_1)}{\varphi(x_2) - \varphi(x_1)} = \frac{f'(\xi)}{\varphi'(\xi)}. \quad (\dagger)$$

In other words,

The average rate of change of a function $f(x)$ with respect to a function $\varphi(x)$ in an interval $[x_1, x_2]$ is equal to the rate of change of $f(x)$ with respect to $\varphi(x)$ at some intermediate "average" point ξ ($x_1 < \xi < x_2$ or $x_1 > \xi > x_2$).

Proof. We suitably generalize the proof of Lagrange's theorem by constructing the auxiliary function

$$F(x) = f(x) - \lambda \varphi(x),$$

where

$$\lambda = \frac{f(x_2) - f(x_1)}{\varphi(x_2) - \varphi(x_1)}.$$

The value of λ is obtained by using the fact that $F(x_1) = F(x_2)$. The function $F(x)$ satisfies all the conditions of Rolle's theorem in the interval $[x_1, x_2]$ †.

Thus there exists a point ξ in the interval (x_1, x_2) such that $F'(\xi) = 0$, i.e.

$$f'(\xi) = \lambda \varphi'(\xi)$$

where $\varphi'(\xi)$ cannot vanish. For, if $\varphi'(\xi) = 0$, then $f'(\xi) = 0$, which is impossible by the conditions of the theorem. Hence, on dividing

* Auguste Cauchy (1789–1859) was a famous French mathematician; his work is fundamental to many branches of modern mathematics.

** The condition that $f'(x)$ and $\varphi'(x)$ have no common zero in the interval (x_1, x_2) (i.e. do not vanish at the same value of x) can be written analytically as

$$[f'(x)]^2 + [\varphi'(x)]^2 \neq 0.$$

† It can easily be verified directly that the values taken by $F(x)$ at the ends of the interval, i.e.

$$F(x_1) = f(x_1) - \frac{f(x_2) - f(x_1)}{\varphi(x_2) - \varphi(x_1)} \varphi(x_1),$$

$$F(x_2) = f(x_2) - \frac{f(x_2) - f(x_1)}{\varphi(x_2) - \varphi(x_1)} \varphi(x_2),$$

are equal.

both sides of the last equation by $\varphi'(\xi)$, we get

$$\lambda = \frac{f'(\xi)}{\varphi'(\xi)}, \quad \text{i.e.} \quad \frac{f(x_2) - f(x_1)}{\varphi(x_2) - \varphi(x_1)} = \frac{f'(\xi)}{\varphi'(\xi)}.$$

This is what we had to prove.

Whereas Lagrange's theorem is concerned with the ratio of the finite increment of a function to the corresponding increment of the independent variable, Cauchy's theorem deals with the ratio of a finite increment of the function to the corresponding increment of another function.

Formula (†) is called the "generalized formula of finite increments" or Cauchy's formula, and the theorem itself the "generalized mean value theorem". In the particular case when $\varphi(x) = x$, Cauchy's formula and theorem reduce to Lagrange's formula and theorem*.

II. A very convenient and important rule for passage to the limit may be proved with the aid of Cauchy's theorem.

L'HÔPITAL'S RULE.** Let functions $f(x)$ and $\varphi(x)$ tend simultaneously to zero or to infinity as $x \rightarrow x_0$ (or as $x \rightarrow \infty$). If the ratio of their derivatives has a limit (which may be infinite), the ratio of the functions themselves has the same limit:

$$\lim \frac{f(x)}{\varphi(x)} = \lim \frac{f'(x)}{\varphi'(x)}.$$

We assume that functions $f(x)$ and $\varphi(x)$ are defined (except possibly for the point x_0 itself) and differentiable in a neighbourhood of x_0 if $x \rightarrow x_0$ (or outside a neighbourhood of zero if $x \rightarrow \infty$). The familiar theorem on the limit of a fraction is inapplicable here.

Proof. Let us take four possible cases.

1. Let $x \rightarrow x_0$, and with this, $f(x) \rightarrow 0$ and $\varphi(x) \rightarrow 0$. We assume that both $f(x)$ and $\varphi(x)$ vanish at $x = x_0$: $f(x_0) = 0$, $\varphi(x_0) = 0$ (for finding the limit of their ratio as $x \rightarrow x_0$, it is of no significance

* Cauchy's theorem cannot be proved, as may appear at first sight, simply by dividing term by term two Lagrange formulae relating to functions f and φ , because generally speaking the intermediate value ξ will not be the same in the two cases.

** L'Hôpital (1661–1704) wrote his "Analysis of infinitesimals" in 1696: this is the first work containing a general exposition of the differential calculus. The rule in question was given there in its simplest form. It was apparently known earlier to J. Bernoulli (1667–1748), a famous Swiss mathematician who taught Euler.

whether or not they are defined at this point, or what their actual values are, if defined). Cauchy's formula now holds for x sufficiently close to x_0 :

$$\frac{f(x)}{\varphi(x)} = \frac{f(x) - f(x_0)}{\varphi(x) - \varphi(x_0)} = \frac{f'(\xi)}{\varphi'(\xi)}, \quad \xi = x_0 + \theta(x - x_0), \quad 0 < \theta < 1.$$

For, since by hypothesis the ratio $f'(\xi)/\varphi'(\xi)$ tends to a limit as $\xi \rightarrow x_0$, there exists a neighbourhood of x_0 such that $f'(x)$ and $\varphi'(x)$ do not have a common zero in it. (Otherwise $f'(\xi)/\varphi'(\xi)$ would have no limit.) The conditions of Cauchy's theorem are therefore fulfilled.

If $x \rightarrow x_0$, then $\xi \rightarrow x_0$ also, and we get:

$$\lim_{x \rightarrow x_0} \frac{f(x)}{\varphi(x)} = \lim_{\xi \rightarrow x_0} \frac{f'(\xi)}{\varphi'(\xi)}$$

or, on changing the notation for the argument on the right-hand side from ξ to x , we arrive at the required equation:

$$\lim_{x \rightarrow x_0} \frac{f(x)}{\varphi(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{\varphi'(x)}.$$

2. Let $x \rightarrow \infty$, and with this, $f(x) \rightarrow 0$ and $\varphi(x) \rightarrow 0$. On setting $x = 1/z$, we arrive at the first case, since if $x \rightarrow \infty$, $z \rightarrow 0$. We have:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{\varphi(x)} = \lim_{z \rightarrow 0} \frac{f\left(\frac{1}{z}\right)}{\varphi\left(\frac{1}{z}\right)},$$

and by what has been proved:

$$\lim_{z \rightarrow 0} \frac{f\left(\frac{1}{z}\right)}{\varphi\left(\frac{1}{z}\right)} = \lim_{z \rightarrow 0} \frac{f'\left(\frac{1}{z}\right)}{\varphi'\left(\frac{1}{z}\right)} \cdot \frac{-\frac{1}{z^2}}{-\frac{1}{z^2}} = \lim_{z \rightarrow 0} \frac{f'\left(\frac{1}{z}\right)}{\varphi'\left(\frac{1}{z}\right)};$$

hence it follows that

$$\lim_{x \rightarrow \infty} \frac{f(x)}{\varphi(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{\varphi'(x)}.$$

3. Let $x \rightarrow x_0$, and with this, $f(x) \rightarrow \infty$ and $\varphi(x) \rightarrow \infty$.

By hypothesis, the limit of the ratio of the derivatives exists; let it be A :

$$\lim_{x \rightarrow x_0} \frac{f'(x)}{\varphi'(x)} = A.$$

We take any point x' sufficiently close to x_0 , say to the right of it: $x' > x_0$ (the proof is the same if we take $x' < x_0$). We have by Cauchy's theorem:

$$\frac{f(x) - f(x')}{\varphi(x) - \varphi(x')} = \frac{f'(\xi)}{\varphi'(\xi)}, \quad \text{where } x_0 < x < \xi < x'.$$

We take out the factor $f(x)$ in the numerator, and $\varphi(x)$ in the denominator, and rewrite the equation as

$$\frac{f(x)}{\varphi(x)} = \frac{f'(\xi)}{\varphi'(\xi)} \frac{1 - \frac{\varphi(x')}{\varphi(x)}}{1 - \frac{f(x')}{f(x)}}. \quad (\dagger)$$

We choose x' so close to x_0 (which involves the required closeness of ξ and x_0) that $f'(\xi)/\varphi'(\xi)$ differs from A by not more than some previously assigned arbitrarily small quantity. This is possible because the ratio in question tends to A , by hypothesis, as $\xi \rightarrow x_0$. After this, we let x approach x_0 so that the ratio

$$\frac{1 - \frac{\varphi(x')}{\varphi(x)}}{1 - \frac{f(x')}{f(x)}}$$

differs from unity by not more than a previously assigned arbitrarily small quantity. This is also possible, since, with the chosen fixed x' , the ratios $\varphi(x')/\varphi(x)$ and $f(x')/f(x)$ tend to zero as x tends to x_0 (because $f(x) \rightarrow \infty$ and $\varphi(x) \rightarrow \infty$). With x sufficiently close to x_0 , the right-hand side of $(\dagger)$ will now differ as little as desired from A , i.e.

$$\lim_{x \rightarrow x_0} \frac{f(x)}{\varphi(x)} = A = \lim_{x \rightarrow x_0} \frac{f'(x)}{\varphi'(x)}.$$

4. Let $x \rightarrow \infty$, and with this, $f(x) \rightarrow \infty$ and $\varphi(x) \rightarrow \infty$. As in case 2, we find by substituting $1/z$ for x that again

$$\lim_{x \rightarrow \infty} \frac{f(x)}{\varphi(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{\varphi'(x)}.$$

None of the above excludes the case of the ratio of the derivatives increasing indefinitely; but in this case, by L'Hôpital's rule (which remains valid), the ratio of the functions themselves also tends to infinity.

L'Hôpital's rule is thus completely proved.

Examples. (1) $\lim_{x \rightarrow 0} \frac{x^3}{x - \sin x}$. The numerator and denominator here tend to zero as $x \rightarrow 0$; they both vanish at $x = 0$, being continuous functions.

We have:

$$\lim_{x \rightarrow 0} \frac{x^3}{x - \sin x} = \lim_{x \rightarrow 0} \frac{(x^3)'}{(x - \sin x)'} = \lim_{x \rightarrow 0} \frac{3x^2}{1 - \cos x};$$

elementary working gives:

$$\lim_{x \rightarrow 0} \frac{x^3}{x - \sin x} = \lim_{x \rightarrow 0} \frac{3x^2}{2 \sin^2 \frac{x}{2}} = \lim_{x \rightarrow 0} 6 \left[\frac{\frac{x}{2}}{\sin \frac{x}{2}} \right]^2 = 6.$$

$$(2) \lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1.$$

$$(3) \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{1+x}}{1} = 1.$$

It will be seen that examples already familiar are now easily solved. But the simplicity is merely apparent in this case, since we have used differentiation of the functions $\sin x$ and $\ln(1+x)$, and this is itself based on a knowledge of the limits of the ratios $\frac{\sin x}{x}$ and $\frac{\ln(1+x)}{x}$ as $x \rightarrow 0$.

It may happen that $f'(x)$ and $\varphi'(x)$ also tend simultaneously to zero or to infinity as $x \rightarrow x_0$ (or as $x \rightarrow \infty$). We obtain on again applying L'Hôpital's rule to their ratio:

$$\lim \frac{f(x)}{\varphi(x)} = \lim \frac{f'(x)}{\varphi'(x)}.$$

It sometimes proves necessary to repeat several times the separate differentiation of the numerator and denominator, and if in the long run $\lim f^{(n)}(x)/\varphi^{(n)}(x)$ exists for a certain n , we can assert the existence of $\lim \frac{f(x)}{\varphi(x)}$ also, where

$$\lim \frac{f(x)}{\varphi(x)} = \lim \frac{f^{(n)}(x)}{\varphi^{(n)}(x)}.$$

$$\begin{aligned} \text{Examples. (1) } \lim_{x \rightarrow 0} \frac{x - x \cos x}{x - \sin x} &= \lim_{x \rightarrow 0} \frac{1 - \cos x + x \sin x}{1 - \cos x} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin x + x \cos x}{\sin x} = \lim_{x \rightarrow 0} \frac{3 \cos x - x \sin x}{\cos x} = 3. \end{aligned}$$

We have applied L'Hôpital's rule three times in succession here.

$$(2) \lim_{x \rightarrow \infty} \frac{\ln x}{x^n} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{n x^{n-1}} = 0, \quad n > 0.$$

$$(3) \lim_{x \rightarrow +\infty} \frac{a^x}{x^n} = \lim_{x \rightarrow \infty} \frac{a^x \ln a}{n x^{n-1}} = \dots = \lim_{x \rightarrow \infty} \frac{a^x (\ln a)^n}{n!} = \infty,$$

if n is a positive integer and $a > 1$.

We have obtained results confirming the properties of the exponential and logarithmic functions which were stated without proof in Chapter I.

An example may be mentioned in which the ratio of the functions has a limit and the ratio of their derivatives does not tend to a limit. For instance,

$$\lim_{x \rightarrow \infty} \frac{x + \sin x}{x} = \lim_{x \rightarrow \infty} \left(1 + \frac{\sin x}{x} \right) = 1,$$

whereas

$$\frac{(x + \sin x)'}{x'} = \frac{1 + \cos x}{1}$$

constantly oscillates between 0 and 2 as $x \rightarrow \infty$, i.e. has no limit.

This example does not contradict L'Hôpital's rule—the rule is simply not applicable to it.

III. The first two cases that we considered, of finding the limit of $f(x)/\varphi(x)$ when both $f(x) \rightarrow 0$ and $\varphi(x) \rightarrow 0$ can be written conventionally as the case $0/0$; the second two cases when $f(x) \rightarrow \infty$ and $\varphi(x) \rightarrow \infty$, as the case ∞/∞ . Of course these notations ($0/0$ and ∞/∞) have no sort of mathematical meaning; they serve merely as the briefest way of indicating a definite case of finding the limit of a function*.

* However, the tradition is still preserved in some text-books of taking $0/0$ and ∞/∞ as so-called "indeterminate forms". Finding the limits of these expressions is called "discovering an indeterminacy", so that L'Hôpital's rule is otherwise called "the rule for discovering an indeterminacy."

It is very often possible to use L'Hôpital's rule to find the limits of functions in cases other than $0/0$ and ∞/∞ . We take the cases: (a) $0 \cdot \infty$, (b) $\infty - \infty$, (c) 1^∞ , (d) ∞^0 , (e) 0^0 .

These notations indicate that, for a given change of the independent variable, the limit is sought of a function expressed respectively as

(a) the product of a function tending to zero and a function tending to infinity;

(b) the difference between two functions tending to positive infinity;

(c) a power whose base tends to 1 and index to infinity;

(d) a power whose base tends to ∞ and index to zero;

(e) a power in which both base and index tend to zero.

There is no need to describe the methods which enable each of these cases to be reduced to the "principal" cases: $0/0$ and ∞/∞ . The reader will recognize the methods from the examples discussed below.

We merely remark that, in cases (c), (d), (e), we first take logarithms of the functions, i.e. first find the limit of the logarithm instead of the function itself, then obtain the limit of the function from the limit of its logarithm (which is permissible because of the continuity of the logarithmic function).

Examples. (a) $A = \lim_{x \rightarrow 0} x^n \cdot \ln x$, $n > 0$; the case $0 \cdot \infty$. We transform to

$$A = \lim_{x \rightarrow 0} \frac{\ln x}{\frac{1}{x^n}};$$

this gives us the case ∞/∞ . We apply L'Hôpital's rule:

$$A = \lim_{x \rightarrow 0} \frac{\frac{1}{x}}{\frac{-n}{x^{n+1}}} = \lim_{x \rightarrow 0} \left(-\frac{x^n}{n} \right) = 0.$$

(b) $A = \lim_{x \rightarrow 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$; the case $\infty - \infty$. We transform to

$$A = \lim_{x \rightarrow 1} \frac{x \ln x - x + 1}{(x-1) \ln x};$$

this is the case $0/0$. We apply L'Hôpital's rule:

$$A = \lim_{x \rightarrow 1} \frac{\ln x}{\frac{x-1}{x} + \ln x} = \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{\frac{1}{x^2} + \frac{1}{x}} = \frac{1}{2}.$$

(c) $A = \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^{1/x^2}$; the case 1^∞ . We transform to

$$\ln A = \ln \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^{\frac{1}{x^2}} = \lim_{x \rightarrow 0} \frac{\ln \frac{\sin x}{x}}{x^2};$$

this is the case $0/0$. We apply L'Hôpital's rule:

$$\ln A = \lim_{x \rightarrow 0} \frac{(\ln \sin x - \ln x)'}{(x^2)'} = \lim_{x \rightarrow 0} \frac{\frac{\cos x}{\sin x} - \frac{1}{x}}{2x}$$

and further

$$\begin{aligned} \ln A &= \frac{1}{2} \lim_{x \rightarrow 0} \frac{-x \sin x}{2x \sin x + x^2 \cos x} \\ &= \frac{1}{2} \lim_{x \rightarrow 0} \frac{-\sin x}{2 \sin x + x \cos x} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{-\cos x}{3 \cos x - x \sin x} = - \end{aligned}$$

whence

$$A = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}.$$

(d) $A = \lim_{x \rightarrow 0} (\cot x)^{1/\ln x}$; the case ∞^0 . We transform to

$$\ln A = \ln \lim_{x \rightarrow 0} (\cot x)^{\frac{1}{\ln x}} = \lim_{x \rightarrow 0} \frac{\ln \cot x}{\ln x};$$

this is the case ∞/∞ . We apply L'Hôpital's rule:

$$\ln A = \lim_{x \rightarrow 0} \frac{\frac{1}{\cot x \sin^2 x}}{\frac{1}{x}} = \lim_{x \rightarrow 0} \frac{-x}{\cos x \cdot \sin x} = -1$$

whence $A = 1/e$.

(e) $A = \lim_{x \rightarrow 0} x^x$; the case 0^0 . We transform to

$$\ln A = \ln \lim_{x \rightarrow 0} x^x = \lim_{x \rightarrow 0} x \cdot \ln x;$$

this is the now familiar case $0 \cdot \infty$. We have

$$\ln A = \lim_{x \rightarrow 0} \frac{\ln x}{\frac{1}{x}}$$

and further, by L'Hôpital's rule,

$$\ln A = \lim_{x \rightarrow 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0,$$

whence $A = 1$.

74. Asymptotic variation of functions and the asymptotes of curves.

I. To describe the behaviour of a function, it is also important to know how it varies as $x \rightarrow \infty$. The characteristic of this variation is given by a comparison of the given function $f(x)$ with some other well-known function $\varphi(x)$ as $|x|$ increases indefinitely. We shall only take as comparison function the linear function $\varphi(x) = ax + b$. The comparison implies geometrically juxtaposing the infinite branch of the graph of $y = f(x)$ with the straight line $y = ax + b$. We introduce in this connection the general geometrical definition of straight asymptote.

Definition. A straight line Γ is called an asymptote of a curve L if the distance of a point of L to the line Γ tends to zero as the point moves away indefinitely from the origin.

(Asymptotes need not be straight lines, though we shall only be concerned with those that are, and their straightness will not be explicitly stated.)

A knowledge of the asymptotes of the infinite branches of the graph of $y = f(x)$ is necessary for a correct representation of the shape of the whole graph, i.e. for seeing how a function varies throughout its domain of definition.

We have to distinguish between the cases of vertical and inclined asymptotes.

(1) Let the curve $y = f(x)$ have a vertical asymptote. The equation of such an asymptote will be $x = x_0$, so that, by the definition of asymptote, we must have $f(x) \rightarrow \infty$ as $x \rightarrow x_0$, and vice versa.

Thus,

If $f(x)$ tends to infinity as $x \rightarrow x_0$, the curve $y = f(x)$ has an asymptote $x = x_0$.

The disposition of the infinite branch of the curve relative to its vertical asymptote $x = x_0$ is found by investigating the sign of the infinity ($\pm \infty$) to which $f(x)$ tends as x tends to x_0 from the right and left. This is clear from Fig. 88, where the possible cases are illustrated.

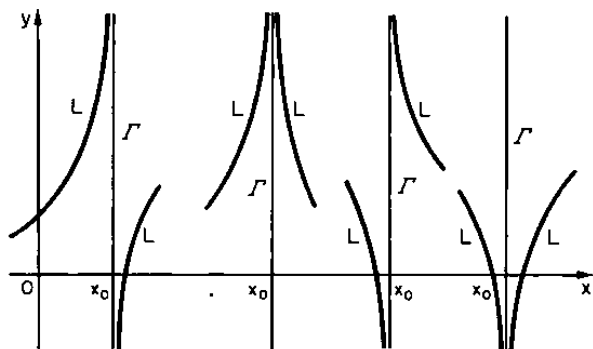


FIG. 88

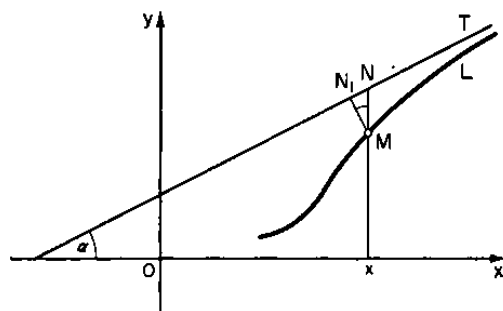


FIG. 89

(2) Let the curve $y = f(x)$ have an inclined asymptote. The equation of such an asymptote is $y = ax + b$. By the definition of asymptote, the distance MN_1 of point M on curve L from the asymptote (Fig. 89) tends to zero as $x \rightarrow \infty$. But the difference MN of the ordinates of point M and the corresponding (i.e. with the same abscissa) point N on the straight line T also tends to zero (and conversely, if $MN \rightarrow 0$, then $MN_1 \rightarrow 0$ also). For we have from Fig. 89:

$$\overline{MN} = \frac{\overline{MN_1}}{\cos \alpha}, \quad \overline{MN_1} = \overline{MN} \cos \alpha,$$

where α is the angle between the asymptote and Ox .

Thus

$$\lim_{x \rightarrow \infty} [f(x) - (ax + b)] = 0.$$

In accordance with this condition, an analytic definition of inclined asymptote can be given.

Definition. If

$$\lim_{x \rightarrow \infty} [f(x) - (ax + b)] = 0$$

the straight line $y = ax + b$ is called an *asymptote to the curve* $y = f(x)$; we say in this case that function $f(x)$ *tends asymptotically to the function* $ax + b$ (or $f(x)$ *is asymptotically equal to* $ax + b$)^{*}.

The greater $|x|$, in general the more correct, i.e. the smaller the relative and absolute error involved on replacing the given function $f(x)$ by its asymptote $y = ax + b$; this, of course, considerably simplifies the use of function $f(x)$.

Thus the problem of the existence and nature of the inclined asymptotes to the curve $y = f(x)$ reduces to considering the existence and values of numbers a and b such that

$$\lim_{x \rightarrow \infty} [f(x) - ax - b] = 0.$$

Hence

$$\lim_{x \rightarrow \infty} x \left[\frac{f(x)}{x} - a - \frac{b}{x} \right] = 0$$

i.e.

$$\lim_{x \rightarrow \infty} \left[\frac{f(x)}{x} - a - \frac{b}{x} \right] = 0 \quad \text{i.e.} \quad \lim_{x \rightarrow \infty} \frac{f(x)}{x} = a. \quad (\dagger)$$

We then find, from the basic condition:

$$\lim_{x \rightarrow \infty} [f(x) - ax] = b. \quad (\dagger\dagger)$$

If equation $(\dagger\dagger)$ holds for any a , $(\dagger)$ also necessarily holds.

^{*} In general, a function $y = f(x)$ tends asymptotically to a function $y = \varphi(x)$ (not necessarily linear) if

$$\lim_{x \rightarrow \infty} [f(x) - \varphi(x)] = 0,$$

or, as is sometimes stated in more subtle branches of analysis, if

$$\lim_{x \rightarrow \infty} \frac{f(x)}{\varphi(x)} = 1.$$

The converse is untrue: equation (†) may hold, yet (††) not, in which case the curve $y = f(x)$ has no asymptote.

For instance, we have for the curve $y = x + \ln x$:

$$\lim_{x \rightarrow \infty} \frac{x + \ln x}{x} = 1, \quad \text{i.e. } a = 1,$$

but

$$\lim_{x \rightarrow \infty} (x + \ln x - 1 \cdot x) = \infty$$

i.e. no asymptote exists in this case.

Thus,

If $f(x)/x$ tends to a finite limit a as $x \rightarrow \infty$ and if $f(x) - ax$ tends to a finite limit b as $x \rightarrow \infty$, the curve $y = f(x)$ has an asymptote $y = ax + b$.

Let $f(x)$ tend to a finite limit ($= b$) as $x \rightarrow \infty$; then obviously $a = 0$ and the curve $y = f(x)$ has an asymptote parallel to Ox , namely $y = b$.

The asymptotic variation of a function can be different according as x tends to positive and negative infinity, so that the cases $x \rightarrow +\infty$ and $x \rightarrow -\infty$ have to be investigated separately.

After finding the asymptote $y = ax + b$, we can find the disposition of the infinite branch of the curve relative to its asymptote by investigating the sign of the expression $\delta = f(x) - ax - b$ as $x \rightarrow \infty$. The branch of the curve either lies above the asymptote ($\delta > 0$) as from a certain position, or below it ($\delta < 0$), or it keeps intersecting the asymptote (δ changes sign an infinite number of times).

Example. Find the asymptotes to the curve $y = x \arctan x$.

Since $y/x = \arctan x$ tends to $\frac{1}{2}\pi$ as $x \rightarrow +\infty$, and tends to $-\frac{1}{2}\pi$ as $x \rightarrow -\infty$, we seek the limits

$$\lim_{x \rightarrow +\infty} \left(y - \frac{\pi}{2} x \right), \quad \lim_{x \rightarrow -\infty} \left(y + \frac{\pi}{2} x \right).$$

We find by L'Hôpital's rule:

$$\lim_{x \rightarrow \pm \infty} \left(y \mp \frac{\pi}{2} x \right) = \lim_{x \rightarrow \pm \infty} \frac{\arctan x \mp \frac{\pi}{2}}{\frac{1}{x}} = \lim_{x \rightarrow \pm \infty} \frac{\frac{1}{1+x^2}}{-\frac{1}{x^2}} = -1.$$

We thus have two asymptotes:

$$y = \frac{\pi}{2}x - 1 \quad \text{and} \quad y = -\frac{\pi}{2}x - 1.$$

Having investigated the curve at finite positions, we can draw the complete curve correctly (Fig. 90). Both asymptotes lie below the curve, since $y - (\frac{1}{2}\pi x - 1) > 0$ for $x > 0$ and $y - (-\frac{1}{2}\pi x - 1) > 0$ for $x < 0$. Since $\lim_{x \rightarrow +\infty} y' = \frac{1}{2}\pi$ and $\lim_{x \rightarrow -\infty} y' = -\frac{1}{2}\pi$ in addition*, the function $x \arctan x$ can be reckoned linear for sufficiently large $|x|$ (with as small an error as desired in defining the function itself and its derivative):

$$x \arctan x \approx \frac{\pi}{2}x - 1 \quad \text{for} \quad x > 0,$$

$$x \arctan x \approx -\frac{\pi}{2}x - 1 \quad \text{for} \quad x < 0.$$

II. We now show that, if the tangent to the curve $y = f(x)$ tends to a limiting position as the point of contact moves away indefinitely, it coincides in this position with the asymptote. In fact, let X and Y denote the current co-ordinates

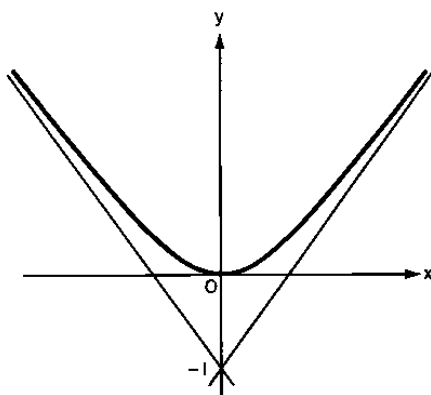


FIG. 90

* If $y = ax + b$ is an asymptote to $y = f(x)$, whilst $f'(x)$ does not tend to a as $x \rightarrow \infty$, then, although the values of $f(x)$ (for large $|x|$) are close to the values of $ax + b$, $f(x)$ cannot be regarded as approximating to the linear function $ax + b$, since its rate of change does not tend to a constant a . Geometrically, the curve $y = f(x)$ can be as close as desired in position to the straight line $y = ax + b$, but it cannot be regarded as approximating to a straight line, if its "direction" does not tend to the direction of this line (the asymptote).

of the tangent at (x, y) to $y = f(x)$; we have

$$Y - y = f'(x)(X - x),$$

i.e.

$$Y = f'(x)X + (y - xf'(x)).$$

The existence of a "limiting position" of this line as $x \rightarrow \infty$ implies that the coefficients of the last equation have limits, which we shall write as a' and b' respectively:

$$\lim_{x \rightarrow \infty} f'(x) = a',$$

$$\lim_{x \rightarrow \infty} [y - xf'(x)] = b'.$$

It remains to show that $a' = a$ and $b' = b$, where a and b are the coefficients of the equation of the asymptote: $y = ax + b$. But $a = \lim_{x \rightarrow \infty} \frac{f(x)}{x}$, which, by L'Hôpital's rule, gives $a = \lim_{x \rightarrow \infty} f'(x)/1 = a'$; $b = \lim_{x \rightarrow \infty} (y - ax)$, which also by L'Hôpital's rule, gives $b = \lim_{x \rightarrow \infty} \left(\frac{y}{x} - a \right) \left(\frac{1}{1/x} \right) = \lim_{x \rightarrow \infty} \left(\frac{xy' - x}{x^2} \right) \left(-1/x^2 \right) = \lim_{x \rightarrow \infty} (y - xy') = b'.$

The converse is untrue, however: a curve may have an asymptote whilst at the same time there is no limiting position of the tangent. Thus, the curve $y = \frac{\sin x^2}{x}$ has an asymptote $y = 0$, whilst the tangent to it may be seen to oscillate constantly as the point of contact becomes indefinitely remote, i.e. the tangent has no limiting position.

We can say as a matter of convention that a straight line occupying the "limiting position" of the tangent as $x \rightarrow \infty$ is the tangent at infinity to the curve.

We have thus shown that the tangent at infinity is an asymptote; it is only when the tangent at infinity exists that we can regard the curve as asymptotically a straight line, and the function as linear.

This proposition enables us to give a simple method for finding the asymptotes of an algebraic curve; an n -th order curve is given by the equation

$$a_0 y^n - a_1(x) y^{n-1} + \dots + a_k(x) y^{n-k} + \dots + a_n(x) = 0, \quad (*)$$

where each $a_k(x)$ is a polynomial of degree not exceeding k ($k = 1, 2, \dots, n$). The abscissae of the points of intersection of the straight line $y = ax + b$ with the curve are obviously found from an equation of degree n in x :

$$a_0(ax + b)^n + a_1(x)(ax + b)^{n-1} + \dots + a_n(x) = 0, \quad (**)$$

having n roots. If the straight line tends to a tangent at infinity (i.e. to an asymptote), at least two points of intersection move away indefinitely whilst merging together, and a correspondingly smaller number of points of intersection remains at a finite distance; the abscissae of the former satisfy the equation which is got from (**) if the coefficients of the two higher powers of x

are equated to zero*. This gives two equations for finding the two parameters (a and b) in the equation of the asymptote.

If curve (*) has a vertical asymptote: $x = a$, the coefficients of the two highest powers of y in the equation

$$a_0 y^n + a_1(a) y^{n-1} + \dots + a_n(a) = 0$$

must similarly vanish. In the case when the equation of degree n contains a term in y^n , i.e. when $a_0 \neq 0$, the curve has no vertical asymptote; when $a_0 = 0$ the second equation $a_1(a) = 0$ or, if $a_1(x), a_2(x), \dots, a_{k-1}(x)$ are identically zero, the equation $a_k(a) = 0$ gives an equation from which the values of a are found.

Let us take as an example the curve

$$y^3 + x^3 - 3\alpha xy = 0,$$

called the *folium of Descartes* (see below, Fig. 92). We have an equation of the third degree, and since $a_0 = 1$, vertical asymptotes are absent. To find the non-vertical asymptotes, we write the equation $(ax + b)^3 + x^3 - 3\alpha x(ax + b) = 0$; on equating to zero the coefficients of the two highest powers of x , we get

$$a^3 + 1 = 0 \quad \text{and} \quad 3a^2 b - 3\alpha a = 0,$$

whence $a = -1$ and $b = -\alpha$. Thus the folium of Descartes has the asymptote

$$y = -x - \alpha.$$

75. General scheme for investigation of functions. Examples. We can now form a scheme in accordance with which functions and curves may be conveniently investigated.

Let us take the function $y = f(x)$. The scheme consists of five sections, in which we establish:

I. (1) the intervals of definition, (2) the points of discontinuity and intervals of continuity, (3) the zeros and intervals of constant sign, (4) the axes and centres of symmetry of the graph (evenness, oddness, periodicity of a function), (5) the domain of the plane in which the complete graph lies.

* In general, if we have an equation

$$\alpha_0 u^n + \alpha_1 u^{n-1} + \dots + \alpha_n = 0$$

with variable coefficients $\alpha_0, \alpha_1, \dots, \alpha_n$ (which, however, remain bounded), a root can only move away to infinity when α_0 tends to zero. For, we know from algebra that

$$u_1 u_2 \dots u_n = (-1)^n \frac{\alpha_n}{\alpha_0},$$

where $u_1, u_2, \dots, u_n$ are the roots of the equation. If $u_1 \rightarrow \infty$, we must have $\alpha_0 \rightarrow 0$; if also $u_2 \rightarrow \infty$, then also $\alpha_1 \rightarrow 0$.

II. (1) the intervals of monotonicity, (2) the extremal points and extremal values.

III. (1) the convex and concave portions of the graph, (2) the points of inflexion.

IV. The asymptotes to the graph.

V. The graph.

A general description is given in the first section of the features of the function and graph, without calling on the resources of analysis.

The concrete nature of the problem which leads to the function being investigated should be borne in mind here. It may happen that it is sufficient to consider a narrower interval of variation of the independent variable than the interval of definiteness of the analytic expression for the function, or that only one branch of the function corresponds to the problem (if many-valued) etc.

It is occasionally convenient to undertake section IV at the same time as the first section, when the general picture of the behaviour of the function is obtained.

Naturally, we can confine ourselves to a description in words of the behaviour of the function, but a correct graphical representation that takes in all its peculiarities is a very economic and expressive means of showing visually the functional relationship. Thus, when working through the first four sections, we should at the same time gradually plot the graph (section V).

Examples. (1) Let us investigate

$$y = \frac{x^2}{1+x}.$$

I. The function is defined and continuous throughout Ox except for the point $x = -1$, where it has an infinite discontinuity. The only zero is $x = 0$.

II. We have:

$$y' = \frac{x^2 + 2x}{(1+x)^2} = \frac{x(x+2)}{(1+x)^2}.$$

The derivative vanishes when $x = 0$ and $x = -2$, being positive in the interval $(-\infty, -2)$, negative in $(-2, 0)$ (except at $x = -1$, where it is not defined), and positive again in $(0, \infty)$. Thus the function is increasing in the first interval, decreasing in the second, and increasing in the third; $x = -2$ is a maximum point, the maxi-

mum value being -4 , whilst $x = 0$ is a minimum point, the minimum value being 0 .

III. We have:

$$y'' = \frac{2}{(1+x)^3}.$$

The second derivative vanishes nowhere, but changes sign from minus to plus as x passes through $x = -1$. There are therefore two intervals in which y'' has constant sign: it is negative in the interval $(-\infty, -1)$, and positive in $(-1, \infty)$. The graph of the function is convex in the first interval and concave in the second.

IV. The straight line $x = -1$ is a vertical asymptote of the graph, where $y \rightarrow +\infty$ as $x \rightarrow -1$ from the right and $y \rightarrow -\infty$ as $x \rightarrow -1$ from the left. Further, since $y/x = x/(1+x)$ tends to 1 as $x \rightarrow \pm\infty$, and $y - x = -x/(1+x)$ tends to -1 as $x \rightarrow \pm\infty$, there is a further asymptote $y = x - 1$. Since the difference

$$\delta = \frac{x^2}{1+x} - (x-1) = \frac{1}{1+x}$$

is positive for $x > -1$ and negative for $x < -1$, the graph is above the asymptote $y = x - 1$ to the right of the straight line $x = -1$, and below it to the left of $x = -1$.

If we take

$$y = \frac{x^2}{1+x} \approx x - 1,$$

the error involved for all $x > 100$, say, does not exceed 0.01 of the value of the function and does not exceed 0.0001 of the value of its rate of change. In practice we can take the linear function $y = x - 1$ as the function $y = x^2/(1+x)$ in the interval $(100, \infty)$, which obviously greatly simplifies its use.

V. On taking into account the results obtained in the first four sections, we can easily draw a graph (Fig. 91) giving a correct representation of the variation of the function throughout Ox . The correctness of our investigation is easily checked in this example, since the curve $y = x^2/(1+x)$ is none other than a hyperbola, as is easily verified by suitably transforming the co-ordinate system.

(2) Let us investigate the function given implicitly by

$$y^3 + x^3 - 3axy = 0.$$

As we know (Sec. 74), the curve given by this equation is called the folium of Descartes.

In cases when the equation for the function does not permit of easy transformation to the explicit form, one should try to find convenient parametric equations. This can sometimes be done by putting $y = tx$, i.e. taking $t = y/x$ as parameter, t being the slope of the straight line from the origin to the current point (x, y) on the curve. We have here:

$$t^3 x^3 + x^3 - 3\alpha t x^2 = 0,$$

whence

$$x = \frac{3\alpha t}{1 + t^3}.$$

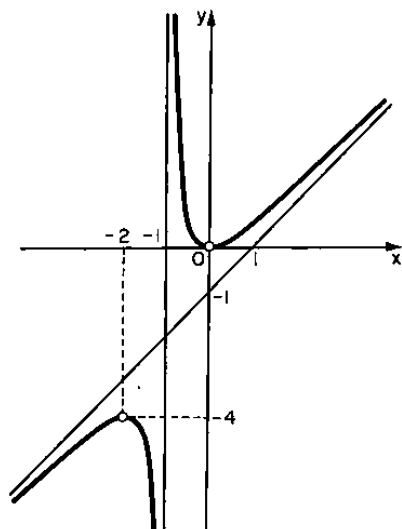


FIG. 91

Thus

$$y = \frac{3\alpha t^2}{1 + t^3}.$$

Our function (and the folium) can therefore be defined by the parametric equations:

$$x = \frac{3\alpha t}{1 + t^3}, \quad -\infty < t < \infty,$$

$$y = \frac{3\alpha t^2}{1 + t^3}.$$

For each value of t there is one corresponding value of x and one of y , i.e. one point (x, y) on the curve.

Since the equation of the curve is unchanged on interchanging x and y , the curve is symmetrical with respect to the bisector of the first and third quadrants.

The co-ordinates x and y become infinite for the same value of t ($t = -1$), so that the curve obviously has no points of discontinuity.

Differentiation gives:

$$x' = 3\alpha \frac{1 - 2t^3}{(1 + t^3)^2}, \quad y' = 3\alpha \frac{2t - t^4}{(1 + t^3)^2}.$$

Variation of t from $-\infty$ to -1 corresponds to the monotonic increase of x (since $x' > 0$) from 0 to ∞ and the monotonic decrease of y (since $y' < 0$) from 0 to $-\infty$. This means that, in the fourth quadrant, the point (x, y) describes an arc OA (Fig. 92) moving away to infinity and having, as we saw in Sec. 74, the asymptote $x + y = -\alpha$. We have for the slope of the tangent:

$$\frac{dy}{dx} = \frac{2t - t^4}{1 - 2t^3}$$

which tends to $-\infty$ as $t \rightarrow -\infty$. We conclude from this that the tangent to arc OA at the point O is the negative semi-axis Oy .

Increase of t from -1 to 0 leads to motion of the point (x, y) along the arc $A'O$, symmetrical to OA with respect to the bisector BB' . Further increase of t from 0 to 1 leads at first (until $t = 1/\sqrt[3]{2}$) to increase of x from 0 to $\sqrt[3]{4}\alpha$, then to decrease to $3\alpha/2$ and to monotonic increase of y from 0 to $3\alpha/2$. The point (x, y) describes the arc $OCDEO$. Variation of t from 1 to ∞ corresponds to movement of the point (x, y) along the arc DEO , symmetrical with respect to BB' with arc $OCDEO$. At $t = 0$ we have $dy/dx = 0$, which implies that the tangent to arc $OCDEO$ at the point O is Ox .

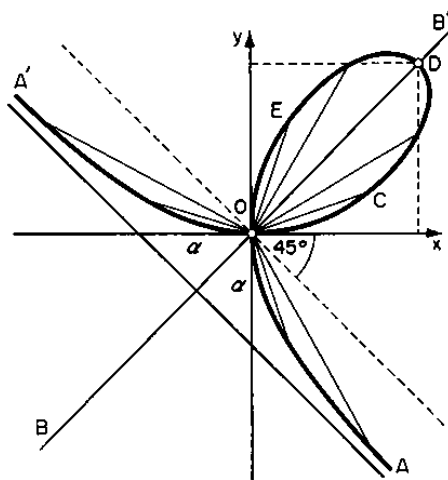


FIG. 92

The form of the curve and the picture of the behaviour of the corresponding function are perfectly clear thanks to the above analysis and the asymptote found in Sec. 74 (see Fig. 92). Actually, the asymptote is easily found here by the general rule, by starting from the parametric equations.

Thus, when the parameter t increases continuously from $-\infty$ to ∞ , the point (x, y) moves along the folium of Descartes in this order: $OAA'OCDEO$, where the branch $OCDEO$ corresponds to variation of t from 0 to ∞ .

76. Solution of equations. Multiple roots. The investigation of a function $f(x)$ generally requires the solution of the equations $f(x) = 0$, $f'(x) = 0$, $f''(x) = 0$; but in their turn, the methods of investigation of functions can aid in finding the roots of an equation.

We remark first of all that a knowledge of the behaviour of the function $y = f(x)$ enables us to establish how many times its graph intersects or touches Ox , i.e. how many real roots the equation $f(x) = 0$ has, and to indicate approximately the places on Ox where the intersections or contacts occur; in other words, it enables us to distinguish the intervals of Ox inside each of which there is only *one* root of the equation $f(x) = 0$. This marking out of the intervals is called *isolating the roots*, the intervals themselves being intervals of isolation.

We shall always start from the assumption that we have been able, in the above or some other way, to isolate the root x_0 of the equation $f(x) = 0$ in some interval $[x_1, x_2]$. Each of the numbers x_1 and x_2 can be reckoned an approximate value of the root x_0 : the first x_1 , being on the low side, and the second, x_2 , on the high side (if $x_1 < x_2$), whilst the difference $x_2 - x_1$ is clearly the maximum error in these approximate values. The methods here described for the approximate solution of equations consist in indicating the means whereby, given an interval of isolation $[x_1, x_2]$ and a function $f(x)$, a new interval $[x'_1, x'_2]$ is found such that

$$x_1 \leq x'_1 < x_0 < x'_2 \leq x_2,$$

i.e. the interval of isolation is compressed; it is evident that x'_1 and x'_2 are in general better approximations for x_0 than x_1 and x_2 .

On applying the same or some other method to the interval $[x'_1, x'_2]$, we obtain still better approximations x''_1, x''_2 to x_0 .

We shall give the three simplest and most convenient methods: "trial and error", "the chord method" or, as it is sometimes called, "regula falsi" (rule of false position) and "the tangent method", which is also known as "Newton's method". There are also combinations of these three methods.

We now bring in the concept of multiple root.

Definition. If a function $f(x)$ can be written in the neighbourhood of a point $x = x_0$ in the form $(x - x_0)^k f_1(x)$, where k is a positive integer and $f_1(x_0) \neq 0$, x_0 is called a *k-ple root* of the equation $f(x) = 0$ (or zero of $f(x)$); it is also said that x_0 is a *root (zero)* of multiplicity k .

If the function $f(x)$ has k derivatives at the point x_0 , it is easily seen by successive differentiation of $f(x) = (x - x_0)^k f_1(x)$ that:

$$f(x_0) = f'(x_0) = \dots = f^{(k-1)}(x_0) = 0, \quad \text{whilst} \quad f^{(k)}(x_0) \neq 0.$$

Conversely, if these last conditions hold, it will be shown in Sec. 79 that $f(x)$ can be written as $(x - x_0)^k f_1(x)$, where $f_1(x_0) \neq 0$, i.e. $x = x_0$ is a k -ple root of the function $f(x)$.

Therefore:

If a function $f(x)$ is k times differentiable, the necessary and sufficient conditions for $x = x_0$ to be a k -ple zero are

$$f(x_0) = f'(x_0) = \dots = f^{(k-1)}(x_0) = 0, \quad f^{(k)}(x_0) \neq 0.$$

If $k = 1$, the root (zero) is described as *single* or *simple*.

For instance, $x = 2$ is a double root, and $x = 1$ a single root of the equation $f(x) = x^3 - 5x^2 + 8x - 4 = 0$, since $f(2) = 0$, $f'(2) = 0$, $f''(2) \neq 0$, whilst $f(1) = 0$, $f'(1) \neq 0$.

Let us indicate a general inequality for the approximate value x' of the root x_0 of the equation $f(x) = 0$, obtained by any method (on the assumption that $f(x)$ has a continuous derivative $f'(x)$ and the root x_0 is simple, i.e. $f'(x_0) \neq 0$). By the mean value formula (Sec. 65), we have

$$f(x') - f(x_0) = f'(\xi)(x' - x_0),$$

whence (bearing in mind that $f(x_0) = 0$)

$$|x' - x_0| = \frac{|f(x')|}{|f'(\xi)|};$$

if $|f'(x)| \geq m \neq 0$ in the initial interval of isolation, then

$$|x' - x_0| \leq \frac{|f(x')|}{m}.$$

This gives the maximum error ε of the approximation x' of the simple root x_0 of the equation $f(x) = 0$ as:

$$\varepsilon = \frac{|f(x')|}{m}.$$

This maximum error is not always satisfactory, however: it may be greater than the length of the interval of isolation.

I. TRIAL AND ERROR. This method is simple, but is not the best of the methods of successive approximation to the root of an equation. Let the interval $[x_1, x_2]$ be an interval of isolation of a root

of the equation $f(x) = 0$ ($f(x)$ is a continuous function) and $f(x_1) \cdot f(x_2) < 0$. The latter condition means that the values of $f(x)$ at the ends of the interval have different signs; suppose for definiteness that $f(x_1) < 0$, $f(x_2) > 0$. We take any value $x = x'$ in the interval $[x_1, x_2]$ and "try" it by substituting it in $f(x)$; if $f(x') < 0$, we obtain on replacing x_1 by x' a narrower interval of isolation x', x_2 ; whereas if $f(x') > 0$, we arrive at a narrower interval of isolation $[x_1, x']$ by replacing x_2 by x' .

On applying trial and error indefinitely, we get a sequence of points $x', x'', \dots$, which may be shown to have the limit x_0 (the root). In view of this, an approximate value for the root can be found to any accuracy by trial and error.

It is convenient to take as the new "trial" point x' the midpoint of the previous interval of isolation: $x' = \frac{1}{2}(x_1 + x_2)$.

Example. We take the equation

$$f(x) = x^3 + 1.1x^2 + 0.9x - 1.4 = 0.$$

Since $f'(x) = 3x^2 + 2.2x + 0.9 > 0$ for all x , the function $f(x)$ increases monotonically, i.e. its graph only cuts Ox once; also, $f(0) = -1.4$ and $f(1) = 1.6$, so that the equation has a unique real root lying in the interval $[0, 1]$.

Let us find $f(0.5)$ and $f(0.7)$ (we take 0.7 instead of 0.75 to simplify the working):

$$f(0.5) = -0.55, \quad f(0.7) = 0.112.$$

This shows that the interval $[0.5, 0.7]$ is an interval of isolation of the required root. We have further:

$$f(0.6) = -0.25, \quad f(0.65) = -0.076, \quad f(0.67) = -0.002.$$

It is clear that we are approaching the root; it lies in the interval $[0.67, 0.7]$. We now bring the right-hand end of the interval of isolation towards the root. We try 0.68; this gives $f(0.68) = 0.034$.

We have thus found a new interval of isolation $[0.67, 0.68]$, which is 100 times smaller than the original $[0, 1]$. If we take 0.675 as the value of the root, the maximum error is 0.005.

Trial and error most commonly requires lengthier working than the chord and tangent methods described below, because in the former method the choice of each successive, more accurate approximation to the root is largely fortuitous, whereas on the other two methods the choice is purposive.

II. THE CHORD METHOD. The conditions are the same as in the method of trial and error. We join the ends of the arc M_1M_2 (Fig. 93) of the curve $y = f(x)$ corresponding to the interval $[x_1, x_2]$ by the chord M_1M_2 . Obviously, in case (I) the point $x = x'_1$ of intersection of this chord with Ox lies closer to x_0 than x_1 ; starting from the new

compressed interval $[x'_1, x_2]$, we similarly obtain the point x''_1 , which will lie even closer to x_0 than x'_1 . We thus find a sequence of points $x_1, x'_1, x''_1, \dots$, tending from the left (i.e. in increasing values of x) to the unknown root x_0 †. Similarly, in case (II) we also get a sequence of points $x_2, x'_2, x''_2, \dots$, but now tending from the right (i.e. in decreasing values of x) to x_0 .

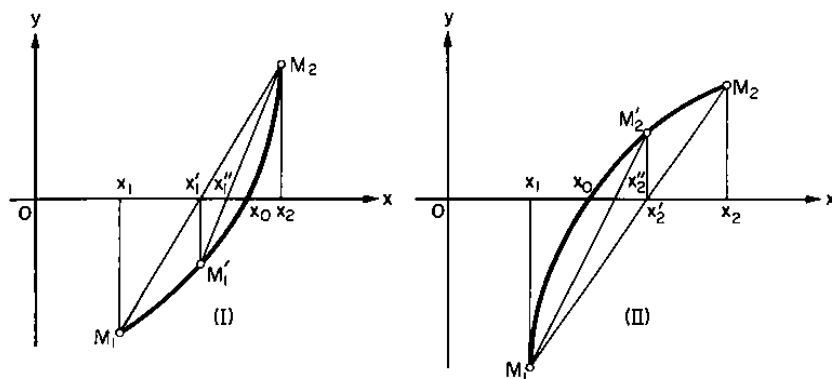


FIG. 93

On writing the equation of the chord M_1M_2 :

$$\frac{y - f(x_1)}{f(x_2) - f(x_1)} = \frac{x - x_1}{x_2 - x_1},$$

we find by putting $y = 0$ the abscissa x' of its point of intersection with Ox :

$$x' = x_1 - f(x_1) \frac{x_2 - x_1}{f(x_2) - f(x_1)} \left(\text{or } x' = x_2 - f(x_2) \frac{x_2 - x_1}{f(x_2) - f(x_1)} \right). \quad (\text{A})$$

This expression, suitable in both cases (I) and (II) (as also when $f(x_1) > 0, f(x_2) < 0$), gives a new approximation x' to the root x_0 in accordance with the two previous approximations x_1 and x_2 . To compress the interval of isolation, x_1 or x_2 must be replaced by x' . We can determine which of the points to replace from our knowledge of the behaviour of $f(x)$ or, if this is difficult, from the sign of $f(x')$.

Example. We apply the chord method to the same equation

$$f(x) = x^3 + 1.1x^2 + 0.9x - 1.4 = 0.$$

Putting $x_1 = 0, x_2 = 1$, we find from formula (A):

$$x' = 1 - f(1) \frac{1 - 0}{f(1) - f(0)} \approx 0.467,$$

† Strictly speaking, this requires proof, which we must omit.

and further, on taking $x_1 = 0.467$, $x_2 = 1$,

$$x'' \approx 0.617$$

similarly,

$$x''' \approx 0.660; \quad x^{IV} \approx 0.668; \quad x^V \approx 0.670;$$

As always in working of this kind, the constancy of the first three decimal places in x^V and x^{VI} shows that we have come close to the true value of the root. To test the accuracy we try 0.671. We have $f(0.671) \approx 0.0012$ and since $f(0.670) < 0$, the new interval of isolation, of total length 0.001, is $[0.670, 0.671]$. If we take the root as approximately 0.6705, the error does not exceed 0.0005, i.e. it is 10 times less than that obtained by trial and error with roughly the same amount of working.

III. THE TANGENT METHOD. Let x_1, x_2 be an interval of isolation of the root x_0 of the equation $f(x) = 0$, such that the corresponding arc of the curve $y = f(x)$ has a tangent at every point and has no point of inflexion (if $f(x)$ is twice differentiable, $f''(x)$ does not change sign in the interval $[x_1, x_2]$). The signs of $f(x_1)$ and $f(x_2)$ may now be the same.

Whereas the chord method is based on replacing the arc by its chord, the tangent method starts from replacement of the arc by its tangent. When $f(x_1) \cdot f(x_2) < 0$, the tangent is drawn at one end M_1 or M_2 of the arc: at the end above Ox if the arc is concave ($f''(x) \geq 0$) (Fig. 94, I) or at the end below Ox , if the arc is convex ($f''(x) \leq 0$) (Fig. 94, II); when $f(x_1) \cdot f(x_2) > 0$, it does not matter at which end of the arc the tangent is drawn.

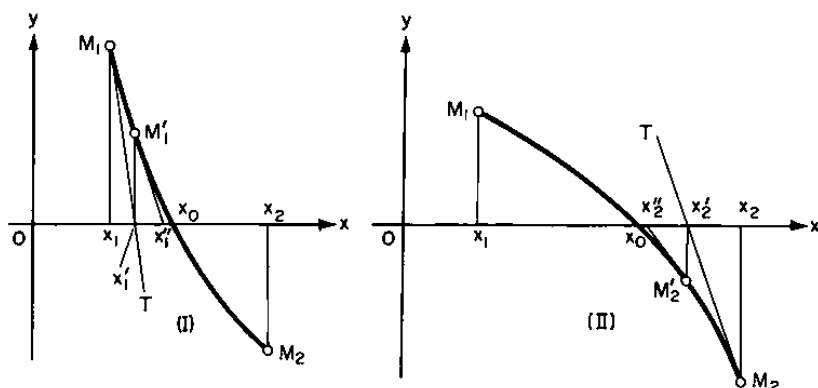


FIG. 94

These conditions guarantee that the point of intersection x'_1 (or x'_2) of the tangent with Ox is always situated between the root x_0 and one end (x_1 or x_2) of the interval of isolation; the interval $[x'_1, x_2]$

or $[x_1, x'_2]$ will be a new compressed interval of isolation of the root x_0^* . Indefinite repetition of the tangent method leads to a sequence having x_0 as its limit. This means that the root can be found to any accuracy by this method.

Figure 94 illustrates two of the six possible cases; the reader should draw the other four cases for himself.

On writing the equation of the tangent M_1T (or M_2T):

$$y - f(x_1) = f'(x_1)(x - x_1) \quad (\text{or } y - f(x_2) = f'(x_2)(x - x_2)),$$

we find by putting $y = 0$ the abscissa x'_1 (or x'_2) of its point of intersection with Ox :

$$x'_1 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad \left(\text{or } x'_2 = x_2 - \frac{f(x_2)}{f'(x_2)} \right). \quad (\text{B})$$

Example. We again take the equation

$$f(x) = x^3 + 1.1x^2 + 0.9x - 1.4 = 0.$$

Since $f''(x) = 6x + 2.2 > 0$ in the interval $[0, 1]$, the tangent is drawn at the point with abscissa 1. By formula (B), we have consecutively:

$$\begin{aligned} x' &= 1 - \frac{f(1)}{f'(1)} \approx 0.738, & x'' &= 0.738 - \frac{f(0.738)}{f'(0.738)} \\ &\approx 0.674, & x''' &\approx 0.671, & x^{IV} &\approx 0.671; \end{aligned}$$

it is clear from the actual method of obtaining the approximations that $f(0.671) > 0$. Since $f(0.670) < 0$, the new interval of isolation of the root will be $[0.670, 0.671]$. This interval has been obtained even more rapidly by the tangent than by the chord method.

IV. COMBINED METHODS. Simultaneous use of the different methods of solution of an equation is sometimes called a *combined method*.

Let us assume fulfilment of the conditions indicated in the above three methods: the arc of the curve $y = f(x)$ corresponding to an interval of isolation $[x_1, x_2]$ of the root x_0 of the equation $f(x) = 0$ has a tangent at every point, has no point of inflexion, and $f(x_1) \cdot f(x_2) < 0$. If, with these conditions, we apply simultaneously say the chord and tangent methods, this will lead to two successive points which tend to x_0 from different sides.

In case (I) of Fig. 94, the tangent method gives an approximation $x_1^{(n)}$ to the root x_0 from the left (on the low side), whilst the chord method gives an approximation $x_2^{(n)}$ from the right (on the high

* If any of the conditions is not fulfilled, the new interval of isolation may be wider than the first, and we move away from the root instead of approaching it.

side); whereas in case (II), vice versa. If the n -th step of the working gives approximations $x_1^{(n)}$ and $x_2^{(n)}$, the next $(n+1)$ -th approximations $x_1^{(n+1)}$ and $x_2^{(n+1)}$ are calculated (say in case (I) of Fig. 94) from the formulae:

$$x_1^{(n+1)} = x_1^{(n)} - \frac{f(x_1^{(n)})}{f'(x_1^{(n)})}; \quad (C)$$

$$x_1^{(n+1)} = x_1^{(n)} - f(x_1^{(n)}) \frac{x_2^{(n)} - x_1^{(n)}}{f(x_2^{(n)}) - f(x_1^{(n)})}. \quad (D)$$

The values found from (C) and (D) approach the root from both sides, which shortens the process of finding the root to a given accuracy.

Naturally, the combined method can also be used by taking the chord or tangent method in conjunction with trial and error, as has in fact already been done above when solving the equation

$$f(x) = x^3 + 1.1x^2 + 0.9x - 1.4 = 0.$$

Let us apply the combined method to this equation. We have $x'_1 \approx 0.467$ (see II) and $x'_2 \approx 0.738$ (see III).

We find from formulae (C) and (D):

$$x''_1 = 0.467 - \frac{f(0.467)(0.738) - 0.467}{f(0.738) - f(0.467)} \approx 0.658, \quad x''_2 \approx 0.674;$$

$$x'''_1 = 0.658 - \frac{f(0.658)(0.674) - 0.658}{f(0.674) - f(0.658)} \approx 0.670, \quad x'''_2 \approx 0.671.$$

We have reached here, at only the third step, the interval of isolation $[0.670, 0.671]$, which is 1000 times less than the original $[0, 1]$.

5. Taylor's Formula and its Applications

77. Taylor's formula for polynomials. Polynomials are the simplest analytic expressions; they are formed from the independent variable and constants with the aid of the operations of addition and multiplication. Thus the calculation of the values of a polynomial is a simple arithmetical operation, which can be carried out without any auxiliary means. On the other hand, the calculation of other functions specified analytically requires as a rule special tables of values of the basic elementary functions. These tables are made up on the basis of Taylor's formula*, which is of extreme importance for all mathematical analysis, and which enables functions to be written

* Taylor (1685–1731), an English contemporary of Newton.

approximately in the form of polynomials in the independent variable. It thus becomes possible to replace complicated analytic expressions by the simplest of them and to reduce the calculation of the values of every function to arithmetical operations only.

We take a function $f(x)$ and first suppose that it is an n -th degree polynomial. Let us pose the problem of "expanding the polynomial $f(x)$ in powers of $(x - x_0)$ ":

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n.$$

The problem amounts to seeking the unknown coefficients $a_0, a_1, a_2, \dots, a_n$. In each concrete case these numbers can readily be found by purely algebraic means. For, suppose we expand the polynomials on the left and right-hand sides of the equation in powers of x . Since we are dealing with an identity, the coefficients of like powers of x must be equal*. On equating coefficients on both sides, we arrive at a system of $n + 1$ equations with $n + 1$ unknowns $a_0, a_1, \dots, a_n$, which then has to be solved.

Example. Let us expand $f(x) = -3 + x - x^2 + 2x^3$ in powers of $x - 1$. We put

$$-3 + x - x^2 + 2x^3 = a_0 + a_1(x - 1) + a_2(x - 1)^2 + a_3(x - 1)^3,$$

whence

$$\begin{aligned} -3 + x - x^2 + 2x^3 &= (a_0 - a_1 + a_2 - a_3) + (a_1 - 2a_2 + 3a_3)x + \\ &\quad + (a_2 - 3a_3)x^2 + a_3x^3 \end{aligned}$$

i.e.

$$\begin{aligned} a_0 - a_1 + a_2 - a_3 &= -3, & a_1 - 2a_2 + 3a_3 &= 1, \\ a_2 - 3a_3 &= -1, & a_3 &= 2. \end{aligned}$$

We find from this system: $a_3 = 2$, $a_2 = 5$, $a_1 = 5$, $a_0 = -1$. Thus we have identically

$$-3 + x - x^2 + 2x^3 = -1 + 5(x - 1) + 5(x - 1)^2 + 2(x - 1)^3.$$

This method of solving the problem is called *the method of undetermined coefficients*.

* This is easily seen. Let the identity hold:

$$\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n = \beta_0 + \beta_1 x + \cdots + \beta_n x^n;$$

we have to show that $\alpha_0 = \beta_0$, $\alpha_1 = \beta_1$, $\dots$, $\alpha_n = \beta_n$. In fact, on setting $x = 0$, we get $\alpha_0 = \beta_0$, i.e. we have identically: $\alpha_1 x + \cdots + \alpha_n x^n = \beta_1 x + \cdots + \beta_n x^n$. On dividing both sides of this by x and again putting $x = 0$, we get $\alpha_1 = \beta_1$. On proceeding in this way, we get all the required equalities.

Definition. The method of undetermined coefficients is the method whereby a required expression of a previously specified type is written with undetermined coefficients which are then found from the condition that certain identities be satisfied.

This method is quite often employed.

But the problem of finding coefficients $a_0, a_1, \dots, a_n$ can be solved in a general form: they happen to be expressible very simply in terms of the derivatives of the function (polynomial) $f(x)$. Let us find these expressions.

First of all, on putting $x = x_0$ in the identity

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n$$

we get $f(x_0) = a_0$. Next, we take the first derivative of both sides:

$$f'(x) = a_1 + 2a_2(x - x_0) + \dots + na_n(x - x_0)^{n-1}$$

and again put $x = x_0$ in this new identity. We now get: $f'(x_0) = a_1$. We differentiate again:

$$f''(x) = 2a_2 + \dots + n(n-1)a_n(x - x_0)^{n-2}$$

and again put $x = x_0$; this gives $f''(x_0) = 2a_2$.

On continuing these successive differentiations and putting $x = x_0$ each time in the identities obtained, we arrive at the following equations:

$$f'''(x_0) = 3 \cdot 2 \cdot a_3, \quad f^{(IV)}(x_0) = 4 \cdot 3 \cdot 2 \cdot a_4, \dots$$

$$f^{(n)}(x_0) = n(n-1) \dots 3 \cdot 2 \cdot a_n.$$

It is clear that the $(n+1)$ -th derivative of both left and right-hand sides is identically zero.

We thus have finally:

$$a_0 = f(x_0), \quad a_1 = f'(x_0), \quad a_2 = \frac{f''(x_0)}{2!},$$

$$a_3 = \frac{f'''(x_0)}{3!}, \dots, \quad a_n = \frac{f^{(n)}(x_0)}{n!},$$

so that

$$\begin{aligned} f(x) = & f(x_0) + f'(x_0)(x - x_0) + \\ & + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \end{aligned} \quad (A)$$

This formula, giving the polynomial $f(x)$ as a polynomial expanded in powers of the given difference $x - x_0$, is called Taylor's formula for polynomials.

In the example considered above, $x_0 = 1$ and

$$f(1) = (-3 + x - x^2 + 2x^3)_{x=1} = -1, \quad f'(1) = (1 - 2x + 6x^2)_{x=1} = 5, \\ f''(1) = (-2 + 12x)_{x=1} = 10, \quad f'''(1) = (12)_{x=1} = 12,$$

and we get the same expansion as above.

As a further interesting example, let us expand the n -th degree polynomial $f(x) = (1+x)^n$ in powers of x , i.e. in powers of the difference $x - 0$. We have:

$$f(0) = 1, \quad f'(0) = n, \quad f''(0) = n(n-1), \dots, \\ f^{(k)}(0) = n(n-1) \dots (n-k+1), \dots, f^{(n)}(0) = n(n-1) \dots 2 \cdot 1.$$

Taylor's formula gives:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots \\ \dots + \frac{n(n-1) \dots (n-k+1)}{k!}x^k + \dots + x^n.$$

We have obtained the familiar binomial formula for a positive integral power. Thus Newton's binomial formula is a particular case of Taylor's formula.

78. Taylor's formula. Now let $f(x)$ be other than a polynomial; formula (A) of Sec. 77 no longer holds, since its right-hand side is a polynomial and the left-hand side is not. We can still regard (A) as approximate, however, instead of accurate. The remarkable fact is that the error is easily found and estimated, and is fairly small with perfectly natural assumptions. It is precisely in this that the significance of the approximate formula lies.

THEOREM. If a function $f(x)$ has derivatives up to and including order $(n+1)$ in an interval (a, b) , it can be written in the form

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots \\ \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} \quad (\dagger)$$

where $a < x < b$, $a < x_0 < b$ and ξ lies between x_0 and x .

Formula $(\dagger)$ is called *Taylor's formula* of order n for the function (fx) at the point x_0 .

Proof. Let x be any point in the neighbourhood of x_0 and let us write formally:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n + R_n, \quad (\text{B})$$

where the last term R_n is the difference between the value of the function and the corresponding value of the polynomial

$$N_n(x - x_0) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n,$$

which is called Taylor's polynomial of degree n for function $f(x)$ at the point x_0 .

We assign to R_n a form similar to the form of the general term of the Taylor polynomial, viz. we put $R_n = \frac{1}{(n+1)!} D_n \cdot (x - x_0)^{n+1}$ and find an expression for D_n ; hence we establish the expression for R_n also.

We bring in the auxiliary function $F(t)$ of the independent variable t :

$$F(t) = f(x) - \left[N_n(x - t) + \frac{1}{(n+1)!} D_n \cdot (x - t)^{n+1} \right]$$

i.e.

$$F(t) = f(x) - \left[f(t) + f'(t)(x - t) + \frac{1}{2!} f''(t)(x - t)^2 + \cdots \right. \\ \left. \cdots + \frac{1}{n!} f^{(n)}(t)(x - t)^n + \frac{1}{(n+1)!} \cdot D_n (x - t)^{n+1} \right].$$

We have constructed the auxiliary function $F(t)$ by subtracting from $f(x)$ the right-hand side of formula (B), in which x_0 is replaced by the auxiliary variable t everywhere except for the factor D_n in the expression for R_n , which also, generally speaking, depends both on x_0 and on x .

The function $F(t)$ vanishes for $t = x_0$ in accordance with equation (B) and with $t = x$, as may be seen directly from the expression for $F(t)$. By Rolle's theorem, its derivative $F'(t)$ must vanish at an intermediate point $t = \xi$, $F'(\xi) = 0$, where $x_0 < \xi < x$ or $x_0 > \xi > x$. Let us find $F'(t)$:

$$\begin{aligned}
 F'(t) = & - \left[f'(t) + f''(t)(x-t) - f'(t) + \right. \\
 & + \frac{1}{2!} f'''(t)(x-t)^2 - f''(t)(x-t) + \dots \\
 & \dots + \frac{1}{n!} f^{(n+1)}(t)(x-t)^n - \frac{1}{(n-1)!} f^{(n)}(t)(x-t)^{n-1} - \\
 & \left. - \frac{1}{n!} D_n(x-t)^n \right],
 \end{aligned}$$

i.e.

$$F'(t) = \frac{1}{n!} (x-t)^n [D_n - f^{(n+1)}(t)].$$

Thus

$$F'(\xi) = \frac{1}{n!} (x-\xi)^n [D_n - f^{(n+1)}(\xi)] = 0.$$

We can now find the required expression for D_n :

$$D_n = f^{(n+1)}(\xi), \quad \xi = x_0 + \theta(x - x_0), \quad 0 < \theta < 1.$$

Thus

$$R_n = \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x-x_0)^{n+1};$$

consequently,

$$\begin{aligned}
 f(x) = & f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!} f''(x_0)(x-x_0)^2 + \dots \\
 & \dots + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n + \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-x_0)^{n+1}. \quad (C)
 \end{aligned}$$

This is what we wanted to prove.

The last term in this accurate formula differs from the general term of the sum only in taking the value of the derivative at a point ξ , lying between x_0 and x , instead of at x_0 itself.

The term R_n , which is the error in the approximation

$$f(x) \approx N_n(x-x_0)$$

is called the remainder term of Taylor's formula of order n .

Lagrange's formula is a particular case ($n = 0$) of Taylor's formula. If $f(x)$ is a polynomial of degree n , $R_n = 0$ and formula (C) becomes formula (A) of Sec. 77.

Taylor's formula is often taken with $x_0 = 0$. In this case*

$$f(x) = f(0) + f'(0)x + \frac{1}{2!} f''(0)x^2 + \cdots + \frac{1}{n!} f^{(n)}(0)x^n + \\ + \frac{1}{(n+1)!} f^{(n+1)}(\xi)x^{n+1}.$$

The polynomial N_n is here expanded directly in powers of the independent variable.

We put Taylor's formula in a different form by writing $x - x_0$ as Δx ($x = x_0 + \Delta x$):

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + \frac{1}{2!} f''(x_0)\Delta x^2 + \cdots \\ \cdots + \frac{1}{n!} f^{(n)}(x_0)\Delta x^n + \frac{1}{(n+1)!} f^{(n+1)}(\xi)\Delta x^{n+1},$$

whence

$$\Delta f(x_0) = f(x_0 + \Delta x) - f(x_0) \tag{D} \\ = df(x_0) + \frac{1}{2!} d^2 f(x_0) + \cdots + \frac{1}{n!} d^n f(x_0) + \frac{1}{(n+1)!} d^{n+1} f(\xi),$$

where $\xi = x_0 + \theta \Delta x$, $0 < \theta < 1$.

The last formula (D) shows that the increment of any function satisfying the conditions in which Taylor's formula is valid is accurately expressed by a very simple sum of its successive differentials (the "Taylor sum"). These differentials are taken at the initial value $x = x_0$, except for the differential of highest order, which is taken at some intermediate value between x_0 and the incremented value ($x_0 + \Delta x$) of the independent variable.

It is immediately clear from formula (D) that, as $\Delta x \rightarrow 0$, the "Taylor sum" of the differentials:

$$df(x_0) + \frac{1}{2!} d^2 f(x_0) + \cdots + \frac{1}{k!} d^k f(x_0), \quad k \leq n,$$

is an approximate value of the infinitesimal increment $\Delta f(x_0)$ to an accuracy of infinitesimals of order not less than $k + 1$. In particular, it follows that *the differential of any twice differentiable function differs from its increment by an infinitesimal of order not less than two.*

* In this case the formula is called Maclaurin's formula.

79. Some applications of Taylor's formula. Examples.

I. THE BEHAVIOUR OF A FUNCTION. We again suppose that the function $f(x)$ has $n + 1$ successive derivatives in a neighbourhood of the point x_0 , where

$$f'(x_0) = f''(x_0) = \dots = f^{(k-1)}(x_0) = 0, \quad \text{and} \quad f^{(k)}(x_0) \neq 0, \quad k \leq n.$$

Then

$$\Delta f(x_0) = f(x_0 + \Delta x) - f(x_0) = \frac{1}{k!} d^k f(x_0) + \dots + \frac{1}{(n+1)!} d^{n+1} f(\xi).$$

As $\Delta x \rightarrow 0$, the expression on the right is a sum of infinitesimals in which the first term has the lowest order of smallness. When Δx is sufficiently small, the sign of the sum of these infinitesimals of different orders is the same as the sign of the infinitesimal of lowest order, provided the latter does not vanish*, i.e. in our case, it is the same as the sign of $d^k f(x_0)$.

Thus, given our conditions, the increment $\Delta f = f(x_0 + \Delta x) - f(x_0)$ has the sign of the differential $d^k f(x_0) = f^{(k)}(x_0) \Delta x^k$ in a sufficiently small neighbourhood of the point x_0 .

If k is an odd number, Δf will have different signs with $\Delta x > 0$ and $\Delta x < 0$, and two cases are possible:

(1) $f^{(k)}(x_0) < 0$; the value of the function is greater to the left of x_0 than at x_0 itself (since $\Delta f > 0$ for $\Delta x < 0$), whilst to the right it is less than at x_0 (since $\Delta f < 0$ for $\Delta x > 0$);

(2) $f^{(k)}(x_0) > 0$; now, conversely, the value of the function to the left of x_0 is less, and to the right greater, than the value at x_0 itself. In other words, in the first case the function is decreasing at x_0 , and in the second increasing.

Whereas if k is an even number, Δf has the same sign (viz. the sign of $f^{(k)}(x_0)$) for $\Delta x < 0$ or $\Delta x > 0$; in this case the value of the function at x_0 is greater than its values to the left and right, if $f^{(k)}(x_0) < 0$, or less than these values if $f^{(k)}(x_0) > 0$. In other words, in the first case the function has a maximum at x_0 , and in the second a minimum.

* For, let $\beta, \gamma, \dots, \omega$ be infinitesimals of higher order than α , and let α not be zero. Then

$$\alpha + \beta + \gamma + \dots + \omega = \alpha \left(1 + \frac{\beta}{\alpha} + \frac{\gamma}{\alpha} + \dots + \frac{\omega}{\alpha} \right),$$

and since the sum $\frac{\beta}{\alpha} + \frac{\gamma}{\alpha} + \dots + \frac{\omega}{\alpha}$ is an infinitesimal, the quantity in brackets is positive and the sign of $\alpha + \beta + \gamma + \dots + \omega$ is the same as the sign of α .

We have thus arrived at general analytic tests for the behaviour of a function "at a point", which are generalizations of the tests given earlier.

TESTS FOR THE BEHAVIOUR OF A FUNCTION "AT A POINT".

If $f'(x_0) = f''(x_0) = \dots = f^{(k-1)}(x_0) = 0$, whilst $f^{(k)}(x_0) \neq 0$, then:

(1) when k is odd, $f(x)$ has no extremum at x_0 , but is decreasing at x_0 if $f^{(k)}(x_0) < 0$, and increasing if $f^{(k)}(x_0) > 0$ (in particular, with $k = 1$ we get the test of Sec. 64);

(2) if k is even, $f(x)$ has an extremum at x_0 ; a maximum if $f^{(k)}(x_0) < 0$ and a minimum if $f^{(k)}(x_0) > 0$ (in particular, with $k = 2$ we get the test for an extremum of Sec. 64).

Similarly, by considering the difference $\Delta f(x_0) - df(x_0)$, we can arrive with the aid of Taylor's formula (D) (Sec. 78) at a general analytic test for the existence of points of inflexion:

TEST FOR THE EXISTENCE OF POINTS OF INFLEXION. If $f''(x_0) = f'''(x_0) = \dots = f^{(k-1)}(x_0) = 0$, whilst $f^{(k)}(x_0) \neq 0$, when k is odd the curve $y = f(x)$ has a point of inflexion at the point of abscissa x_0 , whilst with k even there is no point of inflexion, the curve being convex in the neighbourhood of this point if $f^{(k)}(x_0) < 0$, and concave if $f^{(k)}(x_0) > 0$ (in particular, with $k = 3$ and with $k = 2$ we get the tests of Sec. 71).

Notice that, if the function itself vanishes along with all its first $k - 1$ derivatives at the point x_0 , we can write $f(x)$ as

$$f(x) = (x - x_0)^k f_1(x),$$

where the function $f_1(x)$ does not vanish at $x = x_0$; in other words, x_0 is a zero of multiplicity k of the function (Sec. 76).

II. TAYLOR'S POLYNOMIAL APPROXIMATIONS. Suppose that the values of $f(x)$ and its first n derivatives are known at the point x_0 , $a \leq x_0 \leq b$, together with the number M_{n+1} by which $f^{(n+1)}(x)$ is bounded (in absolute value) in the interval $[a, b]$:

$$|f^{(n+1)}(x)| \leq M_{n+1}.$$

There now exists a number δ_n , bounding the absolute value of the remainder term in Taylor's formula for any point x of the interval $[a, b]$. In fact:

$$\begin{aligned} |R_n| &= \left| \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x - x_0)^{n+1} \right| \\ &= \frac{1}{(n+1)!} \left| f^{(n+1)}(\xi) \right| |x - x_0|^{n+1} \leq \frac{M_{n+1}}{(n+1)!} (b - a)^{n+1} = \delta_n. \end{aligned}$$

We can thus indicate a polynomial of degree n in the difference $x - x_0$ (the Taylor polynomial $N_n(x - x_0)$), the values of which at any point of the interval $[a, b]$ are approximations to the values of the function with the maximum error

$$\delta_n = \frac{M_{n+1}}{(n+1)!} (b-a)^{n+1}.$$

We say that the polynomial $N_n(x - x_0)$ approximates the function $f(x)$ in the interval $[a, b]$ uniformly, with error δ_n (i.e. with a maximum error δ_n suitable for the entire interval $[a, b]$).

This implies geometrically that the curve $y = f(x)$ and the n -th order parabola $y = N_n(x - x_0)$ differ in position in $[a, b]$ by not more than δ_n , or more precisely: the difference of the corresponding ordinates has an absolute value not exceeding δ_n at any point of the interval $[a, b]$.

If a given function $f(x)$ is replaced by its Taylor polynomial $N_n(x - x_0)$ in an interval $[a, b]$, it becomes possible to use the simplest type of function (a polynomial) instead of a complicated function, and a very simple curve (a parabola) instead of a complicated curve. This possibility is based on a knowledge of the maximum error δ_n , characterizing the replacement. In particular, if we put

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

in the interval $[a, b]$ (i.e. replace the arc of a curve by the corresponding segment of tangent), the maximum error δ_1 will be equal to $\frac{1}{2} M_2(b-a)^2$. This fills up the gap left in Sec. 53 when discussing the replacement of $\Delta f(x)$ by $df(x)$.

Another important form of the problem of polynomial approximations is due to the great Russian mathematician P. L. Chebyshev (see Sec. 4). The point is that the n -th degree Taylor polynomial $N_n(x - x_0)$ is by no means the "best polynomial approximation" (in the ordinary sense) of a given function $f(x)$ in a given interval. As a matter of fact, if we require that the n -th degree polynomial $P_n(x)$ have the "least deviation" in the interval from $f(x)$, i.e. that the greatest value of $|f(x) - P_n(x)|$ be minimal, it does not in general happen that $P_n(x) = N_n(x - x_0)$. Polynomials satisfying this requirement were introduced by Chebyshev and now carry his name. The problems arising in connection with the discovery of Chebyshev polynomials have led to the creation of a new branch of mathematical analysis, entitled "constructive theory of functions". Chebyshev's work has been particularly developed by Soviet mathematicians.

III. EXAMPLES. The method of Taylor polynomial approximations described above leads to an extremely simple means of approximate evaluation of a function. Even the computation of the values of a polynomial is greatly simplified if the degree of the polynomial can be lowered.

For instance, let us evaluate

$$f(x) = 2 + x - 2x^2 - x^5 + x^7 + 2x^{10} - x^{20}$$

with $x = 1.025$. We have by Taylor's formula:

$$\begin{aligned} f(1.025) &= f(1 + 0.025) = f(1) + f'(1) \cdot 0.025 + \frac{1}{2!} f''(1) \cdot 0.025^2 + \\ &+ \frac{1}{3!} f'''(1) \cdot 0.025^3 + \dots + \frac{1}{20!} f^{(20)}(1) \cdot 0.025^{20}. \end{aligned}$$

On confining ourselves to the first four terms, we get

$$f(1.025) \approx 2 - 1 \cdot 0.025 - \frac{182}{2} \cdot 0.025^2 - \frac{5250}{6} \cdot 0.025^3 \approx 1.904.$$

We obtain by direct calculation, with the same rounding-off:

$$\begin{aligned} f(1.025) &= 2 + 1.025 - 2 \cdot 1.025^2 - 1.025^5 + 1.025^7 + \\ &+ 2 \cdot 1.025^{10} - 1.025^{20} \approx 1.903. \end{aligned}$$

It will be seen that Taylor's formula enables us to avoid the labour of raising a number with several decimal places to a high power, whilst keeping the answer correct to three figures.

We now turn to some important examples of polynomial approximations obtained by Taylor's formula, and to examples on evaluating functions approximately.

(1) Let us find Taylor's formula of order n at $x_0 = 0$ for e^x . Since $(e^x)^{(k)} = e^x$ for any k , the formula becomes

$$e^x = 1 + x + \frac{1}{2!} x^2 + \dots + \frac{1}{n!} x^n + \frac{1}{(n+1)!} e^\xi x^{n+1}, \quad \xi = \theta x, \quad 0 < \theta < 1.$$

On considering this expansion in an arbitrary but fixed interval $[-M, M]$, we find the error inequality

$$\frac{1}{(n+1)!} e^\xi |x|^{n+1} \leq \frac{1}{(n+1)!} e^M M^{n+1} = \delta_n.$$

Thus the function e^x can be replaced with a maximum error δ_n in the interval $[-M, M]$ by the polynomial

$$N_n(x) = 1 + x + \frac{1}{2!} x^2 + \dots + \frac{1}{n!} x^n.$$

In particular, putting $x = 1$, we get the value for the number e :

$$e \approx 2 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

to an accuracy of $3/(n+1)!$ With $n = 10$, this expression yields seven correct figures: $e = 2.718281 \dots$

If the maximum error ξ is previously assigned, we can use the inequality

$$\delta_n = \frac{1}{(n+1)!} e^M \cdot M^{n+1} < \delta$$

to find the number of terms n which guarantees the required accuracy.

(2) Let us expand $f(x) = \sin x$ by Taylor's formula at $x_0 = 0$. Since

$$f^{(k)}(x) = \sin\left(x + k \frac{\pi}{2}\right),$$

i.e. $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$, $f^{IV}(0) = 0$, $\dots$, we have

$$\sin x = x - \frac{1}{3!} x^3 + \cdots \pm \frac{1}{(2n-1)!} x^{2n-1} + \frac{1}{(2n)!} \sin(\xi + n\pi) \cdot x^{2n},$$

$$\xi = \theta x, \quad 0 < \theta < 1.$$

We obviously have here, by virtue of the inequality $|\sin x| \leq 1$:

$$\delta_n = \frac{1}{(2n)!} M^{2n},$$

where $[-M, M]$ is the interval in which the formula is considered; therefore we have found the maximum error with which the polynomial

$$N_n(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots \pm \frac{1}{(2n-1)!} x^{2n-1}$$

can replace the function $\sin x$ in the interval $[-M, M]$. With $n = 1, 2, 3$, we get the commonly used approximations for $\sin x$:

$$\sin x \approx x, \quad \sin x \approx x - \frac{x^3}{6}, \quad \sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120}.$$

These three formula are arranged in order of increasing accuracy. The first is already familiar from Chapter II.

Figure 95 compares, in the neighbourhood of $x = 0$, the graph of $y = \sin x$ with the graphs of the approximating polynomials

$$y = x, \quad y = x - \frac{1}{6} x^3, \quad y = x - \frac{1}{6} x^3 + \frac{1}{120} x^5.$$

Similarly, it is easy to show that

$$\cos x \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots \pm \frac{1}{(2n)!}x^{2n}$$

with a maximum error of

$$\delta_n = \frac{1}{(2n+1)!} M^{2n+1}$$

valid throughout the interval $[-M, M]$.

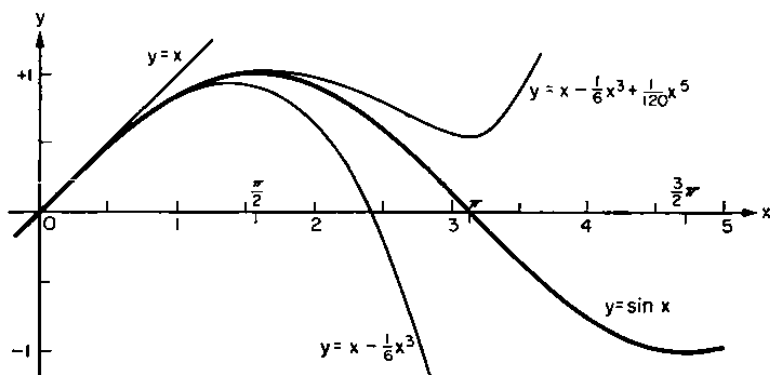


FIG. 95

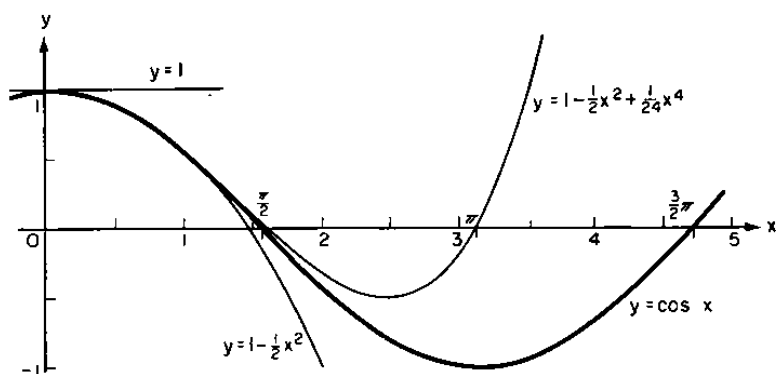


FIG. 96

We obtain with $n = 0, 1, 2$:

$$\cos x \approx 1, \quad \cos x \approx 1 - \frac{x^2}{2}, \quad \cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

In a sufficiently small neighbourhood of $x = 0$, these formulae yield values of $\cos x$ as accurate as desired, the second formula being more

accurate than the first, and the third more accurate than the second. Figure 96 shows for comparison, in the neighbourhood of $x = 0$, the graphs of $y = \cos x$ and of the approximating polynomials

$$y = 1, \quad y = 1 - \frac{x^2}{2}, \quad y = 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

The polynomial approximations for e^x , $\sin x$, $\cos x$ are in fact employed for computing the values of these functions.

(3) Let $f(x) = \ln(1 + x)$. We have:

$$f^k(x) = (-1)^{k-1} \frac{(k-1)!}{(1+x)^k}$$

i.e. $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, $f'''(0) = 2!$, $\dots$, $f^{(k)}(0) = (-1)^{k-1}(k-1)!$

Thus Taylor's formula for $\ln(1+x)^*$ with $x_0 = 0$ becomes

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots + \frac{(-1)^n}{n+1} \cdot \frac{1}{(1+\xi)^{n+1}} x^{n+1},$$

$$\xi = \theta x, \quad 0 < \theta < 1.$$

We obtain the error inequality for the interval $[0, 1]$:

$$|R_n| = \frac{1}{n+1} \frac{x^{n+1}}{(1+\xi)^{n+1}} < \frac{1}{n+1} = \delta_n.$$

This maximum error is too large: to compute the values of $\ln(1+x)$ with sufficient accuracy we should have to take a polynomial of too high a degree. A better formula will be found in Sec. 133, i.e. with a smaller maximum error.

We have the approximate equalities with $n = 1, 2, 3$ ($x > 0$):

$$\ln(1+x) \approx x, \quad \ln(1+x) \approx x - \frac{x^2}{2},$$

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3}.$$

These equalities follow one another in order of increasing accuracy, which is as great as desired with a small enough neighbourhood of $x = 0$. The first equality is familiar from Chapter II.

* Taylor's formula cannot be considered for $\ln x$ with $x_0 = 0$ because this function is not defined at the point $x = 0$.

Figure 97 compares the graphs of $y = \ln(1+x)$ and of its first three Taylor polynomials

$$y = x, \quad y = x - \frac{x^2}{2}, \quad y = x - \frac{x^2}{2} + \frac{x^3}{3}.$$

(4) Finally, we take the binomial $f(x) = (1+x)^m$ with any index m . We find the expansion of $(1+x)^m$ by Taylor's formula of order n with $x_0 = 0^*$. Since

$$f^{(k)}(x) = m(m-1)\dots(m-k+1)(1+x)^{m-k},$$

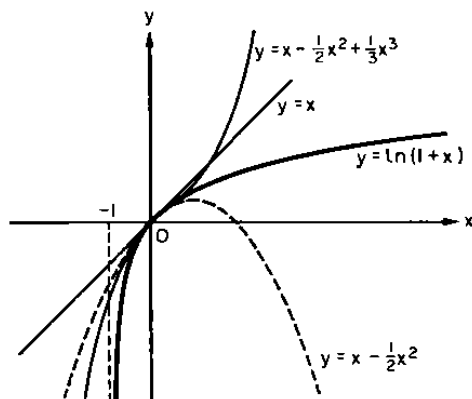


FIG. 97

then $f(0) = 1$, $f'(0) = m$, $f''(0) = m(m-1)$, $\dots$, $f^{(k)}(0) = m(m-1)\dots(m-k+1)$, and we get

$$\begin{aligned} (1+x)^m &= 1 + mx + \frac{m(m-1)}{2!}x^2 + \dots + \\ &+ \dots + \frac{m(m-1)\dots(m-n+1)}{n!}x^n + \\ &+ \frac{m(m-1)\dots(m-n)}{(n+1)!}(1+\xi)^{m-n-1}x^{n+1}. \end{aligned}$$

If m is a positive integer and $n \geq m$, the formula breaks off at the $(m+1)$ -th term: all future terms may easily be seen to vanish, i.e. the remainder term is zero; the formula becomes Newton's binomial formula (see Sec. 77). But if m is not a positive integer,

* We do not take simply the power function x^m because, if m is not a positive integer (and only this case is of interest), the derivatives of x^m , as from a certain order, do not exist at $x = 0$.

Taylor's formula gives a very important approximate expression for $(1+x)^m$ as an n -th degree polynomial with so-called "binomial coefficients".

We shall not estimate the remainder term here — this will be done in Chapter IX.

With $n = 1, 2, 3$, we have the approximations:

$$(1+x)^m \approx 1 + mx, \quad (1+x)^m \approx 1 + mx + \frac{m(m-1)}{2}x^2,$$

$$(1+x)^m \approx 1 + mx + \frac{m(m-1)}{2}x^2 + \frac{m(m-1)(m-2)}{6}x^3.$$

These formulae also have any desired accuracy in a small enough neighbourhood of $x = 0$, the second being more accurate than the first, and the third than the second. We have already encountered the first formula (see Sec. 53).

6. Curvature

80. Curvature. We introduce in this article a new quantitative characteristic of curves, called "curvature", defining the degree to which the curve "bends". We shall assume here that every curve considered has a tangent at any point, which rotates continuously as the point of contact moves continuously along the curve.

We first of all introduce the angle of displacement of an arc of a curve as *the angle φ between the tangents at its ends M_0 and M_1* (Fig. 98).

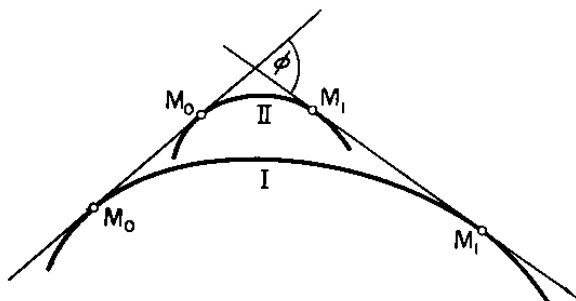


FIG. 98

The angle φ gives some idea of the bending of the arc M_0M_1 , since it is the angle through which the tangent rotates as the point of contact moves from the start M_0 to the end M_1 ; the greater φ , the greater, obviously, the bending of the arc. But two arcs which

clearly bend to different extents may have the same angle φ . For instance, curve I in Fig. 98 has a greater length and a smaller curvature than curve II. In view of this, the curvature is characterized by taking the angle of displacement of the arc per unit length.

Definition. The ratio of the angle of displacement of an arc to its length is called the mean curvature of the arc:

$$K_{av} = \varphi / M_0 M_1.$$

The mean curvature gives a quantitative, but merely general, idea of the curvature of the entire arc $M_0 M_1$. The mean curvature of separate pieces of an arc will in general be different to the mean curvature corresponding to the complete arc. To avoid this indeterminacy, we establish the concept of curvature at a point M_0 , based on the idea that, the smaller the arc $M_0 M_1$, the better the mean curvature will characterize the bending of the curve at the point M_0 .

Definition. The curvature K of a curve at a point M_0 is defined as the limit to which the mean curvature K_{av} of the arc $M_0 M_1$ tends as the end M_1 of the arc approaches its given initial point M_0 :

$$K = \lim_{M_1 \rightarrow M_0} K_{av} = \lim_{M_1 \rightarrow M_0} \varphi / M_0 M_1.$$

In particular, we have for the circle of radius r (Fig. 99):

$$K_{av} = \varphi / M_0 M_1 = \varphi / r \varphi = 1/r;$$

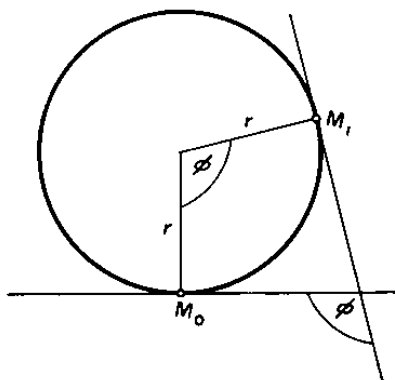


FIG. 99

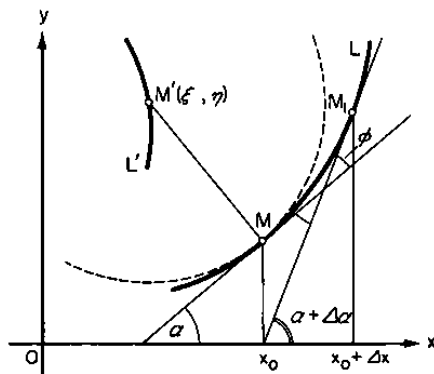


FIG. 100

as we see, the mean curvature of a circle is constant, i.e. its curvature is constant at any point and equal to the reciprocal of the radius.

This is in complete agreement with our immediate idea of the nature of the "bending" of a circle.

Let us now find an expression for the curvature of a curve given by the equation $y = f(x)$ in a system of Cartesian co-ordinates; we assume that the function $f(x)$ is twice differentiable for the values of x considered. We take points M and M_1 on the curve with abscissae x and $x + \Delta x$ respectively. Further, we write α and $\alpha + \Delta\alpha$ for the angles formed with Ox by the tangents at points M and M_1 , and Δs for the length of arc MM_1 . Since the angle φ of displacement of the arc MM_1 is equal to $|\Delta\alpha|$ (Fig. 100), we have

$$K_{av} = \left| \frac{\Delta\alpha}{\Delta s} \right| = \frac{\left| \frac{\Delta\alpha}{\Delta x} \right|}{\left| \frac{\Delta s}{\Delta x} \right|}.$$

As $M_1 \rightarrow M$, $\Delta x \rightarrow 0$ and the required curvature is

$$K = \lim_{\Delta x \rightarrow 0} K_{av} = \frac{\left| \frac{d\alpha}{dx} \right|}{\left| \frac{ds}{dx} \right|} = \frac{\frac{d\alpha}{dx}}{\sqrt{1 + y'^2}}$$

(Fig. 57). We find from the equation $\tan\alpha = y'$, i.e. $\alpha = \arctan y'$,

$$\frac{d\alpha}{dx} = \frac{y''}{1 + y'^2},$$

i.e.

$$K = \left| \frac{y''}{(1 + y'^2)^{3/2}} \right|. \quad (\text{A})$$

The sign of the expression $\frac{y''}{(1 + y'^2)^{3/2}}$ indicates the side towards which the curve is convex "at the point M ", whilst the absolute value gives the curvature, i.e. a measure of the bending of the curve at M .

The curvature is defined by formula (A) as a function of the abscissa of the point of the curve. If the equation is given parametrically as $x = \varphi(t)$, $y = \psi(t)$, we have

$$y' = \frac{\psi'(t)}{\varphi'(t)}, \quad y'' = \frac{\psi'(t)\varphi''(t) - \varphi'(t)\psi''(t)}{[\varphi'(t)]^3}$$

and

$$K = \left| \frac{\psi'(t)\varphi''(t) - \varphi'(t)\psi''(t)}{[\varphi'^2(t) + \psi'^2(t)]^{3/2}} \right|,$$

or more briefly:

$$K = \left| \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}} \right|. \quad (\text{B})$$

The curvature is here a function of parameter t , corresponding to the point of the curve in question. When $t = x$, formula (B) becomes formula (A).

Finally, suppose that the curve is given by an equation in polar co-ordinates $\rho = f(\varphi)$. Taking φ as the parameter, we have from the equations $x = \rho \cos \varphi$, $y = \rho \sin \varphi$:

$$\begin{aligned} x' &= \rho' \cos \varphi - \rho \sin \varphi, & x'' &= \rho'' \cos \varphi - 2\rho' \sin \varphi - \rho \cos \varphi, \\ y' &= \rho' \sin \varphi + \rho \cos \varphi, & y'' &= \rho'' \sin \varphi + 2\rho' \cos \varphi - \rho \sin \varphi. \end{aligned}$$

On substituting these expressions for x' , x'' , y' , y'' in formula (B), we get a formula for the curvature in polar co-ordinates:

$$K = \left| \frac{\rho^2 + 2\rho'^2 - \rho\rho''}{(\rho^2 + \rho'^2)^{3/2}} \right|. \quad (\text{C})$$

The curvature is a local feature relating to a point of the curve. A circle (and straight line) alone have constant curvature; the curvature of a straight line is zero, as is clear say from formula (A), this being once more in complete agreement with our immediate conception of the lack of bending in a straight line. In other curves, the curvature in general varies from point to point.

The curvature $K = \left| \frac{d\alpha}{ds} \right|$ can be regarded as the rate (taking the absolute value) of change of the angle of inclination of the curve with respect to its length.

Finally, it should be noted that the curvature depends neither on the co-ordinate system nor on the position of the curve in the plane, but only on properties inherent in the curve itself.

§1. Radius, centre and circle of curvature. We draw a normal to the curve (Fig. 100) at M towards the concave side and mark off a segment $MM' = r$, equal to the reciprocal of K :

$$r = \frac{1}{K}.$$

Definition. The segment MM' is called the *radius of curvature*, the point M' the *centre of curvature*, and the circle with centre at M' and radius MM' the *circle of curvature* of the curve at its point M .

The circle of curvature obviously has the same curvature as the given curve at M .

We have for the radius of curvature:

$$r = \left| \frac{(1 + y'^2)^{3/2}}{y''} \right| = \left| \frac{(x'^2 + y'^2)^{3/2}}{y''x' - x'y'} \right| = \left| \frac{(\varrho^2 + \varrho'^2)^{3/2}}{\varrho^2 + 2\varrho'\varrho'' - \varrho\varrho''} \right|. \quad (D)$$

The radius of curvature can be used instead of the curvature as a measure of the local bending of a curve; the radius of curvature is the smaller, the greater the bending at the point. It should be noted that the radius of curvature of a straight line at any point is conventionally reckoned "infinite", just as a straight line is taken conventionally to be a circle of "infinite" radius.

Replacement of an infinitesimal arc about the point M by the corresponding segment of tangent involves an infinitesimal maximum error of not lower than the second order, whilst it can be shown that replacing it by the corresponding arc of the circle of curvature involves an infinitesimal error of not lower than the third order. We can therefore regard a small arc of a curve as "almost" the arc of the circle of curvature, with more correctness, i.e. smaller error, than as the segment of tangent.

The concepts introduced thus serve for a more accurate characterization of the curve at the point M . They indicate the degree of bending of the curve by comparing it with the circle having a common tangent and the same curvature at their common point M .

As we know, the first derivative $f'(x)$ has a very simple geometrical interpretation; whereas the second derivative $f''(x)$ cannot be interpreted so simply geometrically. However, as is clear from formula (D), a close connection exists between the second derivative $f''(x)$ and the radius of curvature of the graph of $y = f(x)$ at the corresponding point. In particular, if the radius of curvature does not exist or is equal to infinity, $f''(x)$ does not exist or vanishes for the corresponding value of x . Conversely, at points of the graph for which $f''(x)$ does not exist or $f''(x) = 0$, the radius of curvature does not exist or is infinite. Thus, *at a point of inflexion a curve either has no radius of curvature or its radius of curvature is infinite.*

Let us take the function whose graph is illustrated in Fig. 101. The curve AMB consists of the arcs of two circles C' and C'' . It is immediately clear from the graph that the function has a derivative everywhere in the interval $(-1, 2)$ (the curve has a tangent), which moreover varies continuously. As regards the second derivative, this does not exist at all at the point $x = 0$.

This can be verified by direct calculation. But a simple and visual explanation is possible with the aid of the radius of curvature. For, to the left of point M

the radius of curvature is everywhere equal to 1, and to the right is everywhere equal to 2; thus the radius of curvature does not exist at point M , so that the second derivative likewise does not exist at $x = 0$. Just as a curve has "two tangents" at a double point, the curve AMB has "two radii of curvature" at the point M : one corresponding to arc AM on the left of M , the other to arc MB on the right of M .

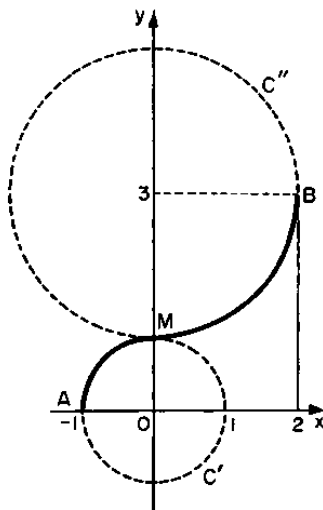


FIG. 101

Let the function $y = f(x)$ be smooth in a given interval, i.e. have a continuous first derivative everywhere. Its graph has a tangent at every point and represents a continuous, smooth (see Sec. 54) curve, along which the tangent "rotates" continuously. But it may happen that the function has no second derivative at certain points, so that the curve has no radius of curvature. It is now impossible to regard the curve as specially smooth from the point of view of variation of the curvature, which latter has discontinuities at points of abutment of pieces with continuously varying curvature. These points of the curve play the same role with respect to curvature as the double points play with respect to its direction.

For instance, the curve AMB (Fig. 101) is not "smooth" in the sense of curvature, because, notwithstanding the apparent smoothness of the curve, its curvature has a discontinuity at the point M — it passes from the value 1 to the value 2 (the radius of curvature at M jumps from a value 1 to a value $\frac{1}{2}$).

If $f(x)$ also has a continuous second derivative everywhere, its graph has continuous curvature at every point, and the curve $y = f(x)$

can be regarded as "smooth" in a stronger sense: not only the tangent, but also the curvature varies continuously along it.

The curve $y = f(x)$ must be regarded as the "smoother", the more derivatives of the function $f(x)$ exist at every point of the interval. It is worth mentioning again that the elementary functions, generally speaking, have derivatives of all orders except at individual points.

These facts are of great importance in certain practical problems, as for instance in the laying of railway lines. As we know from mechanics, when a body moves along a circle of radius R , the centrifugal force P is given by $P = mv^2/R$, where m is the mass of the body, and v its velocity; this force is directed along the radius of the circle. When the motion is along any other trajectory, a force is directed along the normal to it, and is given by the same formula, R being the radius of curvature of the trajectory at the given point. Suppose that the velocity is constant; then the force will have discontinuities at points where the curvature of the trajectory is discontinuous. This explains the jolting of the carriages at bends, although the track usually appears to bend round smoothly. To avoid jolting, attempts are made to arrange the bend so that the curvature varies continuously. For instance, a straight is joined to an arc of a circle by means of a so-called transition curve, which permits of a continuous passage of the curvature from zero to the curvature of the given circle. The cubical parabola $y = ax^3$ is often used as a transition curve. We have for this:

$$y' = 3ax^2, \quad y'' = 6ax \quad \text{and} \quad K = \left| \frac{6ax}{(1 + 9a^2x^4)^{3/2}} \right|.$$

With $x = 0$, $y' = 0$: the parabola touches Ox at the origin and has zero curvature at this point. (This is obvious: the origin is a point of inflexion of the parabola $y = ax^3$.) The curvature then increases up to a certain x . The semi-axis Ox can thus be joined to a chosen arc of a circle by means of a cubical parabola in such a way that the curvature of the resulting curve increases continuously from zero to the curvature of the circle.

If the trajectory is so "smooth" that the curvature exists and is continuous at any point, the motion remains smooth without any change of velocity at bends.

82. Evolute and involute. Let L be a given curve with continuous curvature. For every point M of curve L there is now a corresponding point M' — the centre of curvature of L at M .

Definition. The locus of centres of curvature of a curve L is called its *evolute* L' , whilst curve L itself is an *involute* with respect to its evolute L' .

If the curve L is given by an equation in a system of Cartesian co-ordinates, the co-ordinates ξ , η of the centre of curvature are found as follows. We write down two conditions:

(1) the point M' lies on the normal to the curve:

$$\eta - y = -\frac{1}{y'}(\xi - x);$$

(2) the distance of M' from the point M is equal to the radius of curvature r :

$$(x - \xi)^2 + (y - \eta)^2 = r^2.$$

On solving these two equations for ξ and η and taking into account that $\eta > y$ if $y'' > 0$, and $\eta < y$ if $y'' < 0$, and using the equation for the radius of curvature r , we arrive at the following equations:

$$\xi = x - y' \frac{1 + y'^2}{y''}, \quad \eta = y + \frac{1 + y'^2}{y''}. \quad (\text{A})$$

The co-ordinates ξ, η are given here as functions of the abscissa of the corresponding point of curve L .

If the curve L is given parametrically, we have:

$$\xi = x - y' \frac{x'^2 + y'^2}{x'y'' - y'x''}, \quad \eta = y + x' \frac{x'^2 + y'^2}{x'y'' - y'x''}. \quad (\text{B})$$

The co-ordinates ξ, η are given here as functions of the same parameter in terms of which the co-ordinates x and y are given. In the particular case when $t = x$, formula (B) becomes formula (A).

Thus, knowing the equation of the involute L , we can readily find the parametric equations of the evolute L' from formulae (A) or (B). The converse problem, of defining analytically the involute, given the evolute, is more difficult; it is solved with the aid of the integral calculus. But we can consider here some closely interconnected properties of the evolute and involute.

THEOREM 1. The tangent to the evolute is normal to the involute (Fig. 102).

Proof. The slope of the tangent to the evolute, $d\eta/d\xi$, is found from formula (B) by first transforming (B) to a form more convenient for differentiation. We write

$$\frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \nu$$

so that $|\nu| = r$, where r is the radius of curvature. Now:

$$\xi = x - \frac{y'}{\sqrt{x'^2 + y'^2}} \nu, \quad \eta = y + \frac{x'}{\sqrt{x'^2 + y'^2}} \nu.$$

We now observe that the following equations hold:

$$\left(\frac{y'}{\sqrt{x'^2 + y'^2}}\right)' = \frac{x'}{v} \quad \text{and} \quad \left(\frac{x'}{\sqrt{x'^2 + y'^2}}\right)' = -\frac{y'}{v},$$

so that

$$\frac{d\xi}{dt} = x' - \frac{x'}{v} v - \frac{y'}{\sqrt{x'^2 + y'^2}} v' = -\frac{y'}{\sqrt{x'^2 + y'^2}} v',$$

$$\frac{d\eta}{dt} = y' - \frac{y'}{v} v + \frac{x'}{\sqrt{x'^2 + y'^2}} v' = \frac{x'}{\sqrt{x'^2 + y'^2}} v'$$

and

$$\frac{d\eta}{d\xi} = \frac{d\eta/dt}{d\xi/dt} = -\frac{x'}{y'} = -1/\frac{dy}{dx}.$$

This equality shows that the tangent to the evolute L' (of slope $d\eta/d\xi$) is parallel to the normal to the given curve L (slope $-1/(dy/dx)$); and since, in addition, the normal passes through the corresponding point (the point of contact) $M'(\xi, \eta)$ of the evolute, these straight lines coincide. Hence it follows that the evolute L' is touched by all the normals to the involute or, as is generally said (see Secs. 45, 164), is the envelope of the family of normals. If a large number of normals is drawn to the given curve L , the curve enveloped by them gives an approximate idea of the evolute L' .

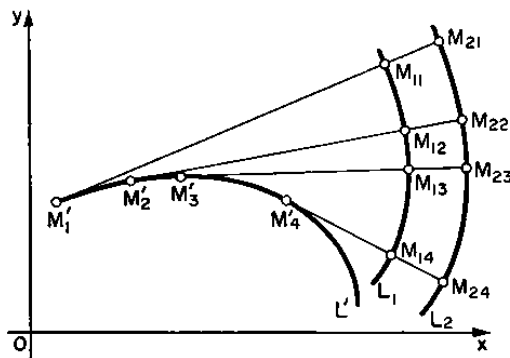


FIG. 102

Let us mark off on each normal to the curve L_1 the same segment $M_{11}M_{21} = M_{12}M_{22} = M_{13}M_{23} = \dots$ (Fig. 102). We obtain a new curve L_2 , "parallel" to the given curve, the normals to which coincide with the normals to L_1 . Thus both these "parallel" curves have the same evolute. Therefore, each curve has one evolute, but the latter has an infinite set of involutes, forming a family of "parallel" curves.

THEOREM 2. The increment of the length of arc of the evolute is equal to the corresponding increment of the radius of curvature of a given involute (assuming that the radius of curvature varies monotonically along the arc of the involute).

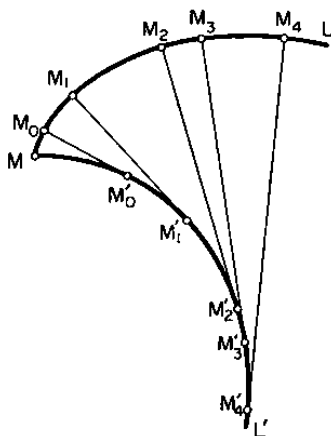


FIG. 103

Proof. We take an arc of the evolute and find an expression for ds/dx . We have by formula (A):

$$\left(\frac{ds}{dx}\right)^2 = \left(\frac{d\xi}{dx}\right)^2 + \left(\frac{d\eta}{dx}\right)^2 = \frac{1+y'^2}{y'^4} [3y'y''^2 - y'''(1+y'^2)]^2.$$

But the reader will readily obtain the same expression for the square of the derivative of the radius of curvature of the involute with respect to the abscissa, i.e. for $(dr/dx)^2$. Thus

$$\left(\frac{ds}{dx}\right)^2 = \left(\frac{dr}{dx}\right)^2, \quad \text{whence} \quad \left|\frac{ds}{dx}\right| = \left|\frac{dr}{dx}\right|.$$

We conclude by Cauchy's formula (Sec. 73)* that

$$|\Delta s| = |\Delta r|.$$

This is what we had to prove.

The above two properties enable a very simple mechanical method of constructing an involute to be given. We imagine a flexible,

* For, if two functions f and φ are such that $|f'| = |\varphi'|$ at every point of an interval, it follows from the formula $|\Delta f/\Delta \varphi| = |f'(\xi)/\varphi'(\xi)|$ that $|\Delta f| = |\Delta \varphi|$. If r does not vary monotonically along the arc, dr/dx vanishes and Cauchy's formula is inapplicable.

inextensible string stretched about the given convex curve (the evolute). We start to unwind the string whilst always keeping it taut. Every point of the string now describes an involute, for which the string is a normal; a fixed point of the string describes in the process of unwinding a definite involute, since the length of the freed part of the string is precisely equal to the length of arc of the evolute along which the part of the string was stretched. In Fig. 103, $M_1M'_1 - M_0M'_0 = M'_0M'_1$, $M_2M'_2 - M_1M'_1 = M'_1M'_2, \dots$, i.e. by the second property, the string segments $M_1M'_1, M_2M'_2, \dots$ are radii of curvature of the same curve L . The unwinding process is equivalent to the motion (rolling) without slip of a straight line along curve L' ; every point of the straight line describes its involute.

It can be seen from geometrical considerations that the points of the curve at which the curvature (or radius of curvature) attains an extremum correspond to double points (cusps) of the evolute. The points of the curve with extremal curvature are called its vertices, this definition of vertex being independent of the system of coordinates and the method of specifying the curve.

83. Examples.

I. INVOLUTE OF CIRCLE. A fixed point of a straight line which rolls without slip along a circle describes an involute of the circle. Let us take the involute which passes through the point $A(a, 0)$ of the circle (Fig. 104). Bearing in mind that $PA = PC$, the equation of the involute is easily obtained:

$$x = a(\cos t + t \sin t), \quad y = a(\sin t - t \cos t),$$

where a is the radius of the circle and t the angle POA .

The curve defined by these parametric equations is in fact called an involute of a circle. Conversely, by formulae (B) of Sec. 82, we find the equations of the evolute of the involute of a circle:

$$\xi = a \cos t, \quad \eta = a \sin t.$$

We see that this is in fact the original circle.

II. EVOLUTE OF PARABOLA AND ELLIPSE. We have for the parabola $y^2 = 2px$:

$$\xi = 3x + p, \quad \eta = -\frac{4x\sqrt{x}}{\sqrt{2p}}.$$

Elimination of x yields the familiar equation of the semi-cubical parabola (see Sec. 21):

$$\eta^2 = \frac{8}{27p} (\xi - p)^3.$$

The evolute of a parabola is thus a semi-cubical parabola (Fig. 105). The minimum radius of curvature of the parabola is equal to p , to which correspond the vertex of the parabola and the cusp of the evolute.

We have for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:

$$\xi = \frac{a^2 - b^2}{a^4} x^3, \quad \eta = \frac{a^2 - b^2}{b^4} y^3.$$

On using the equation of the ellipse and eliminating x and y , we get:

$$\left(\frac{\xi}{b}\right)^{\frac{2}{3}} + \left(\frac{\eta}{a}\right)^{\frac{2}{3}} = \left(\frac{a^2 - b^2}{ab}\right)^{\frac{2}{3}}.$$

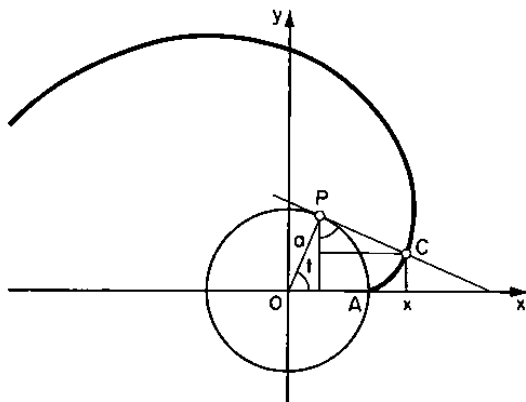


FIG. 104

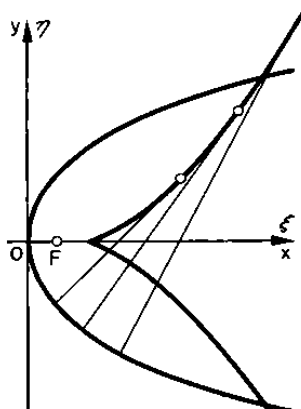


FIG. 105

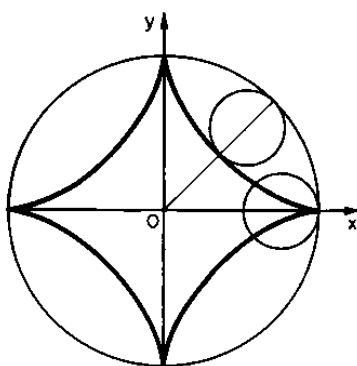


FIG. 106

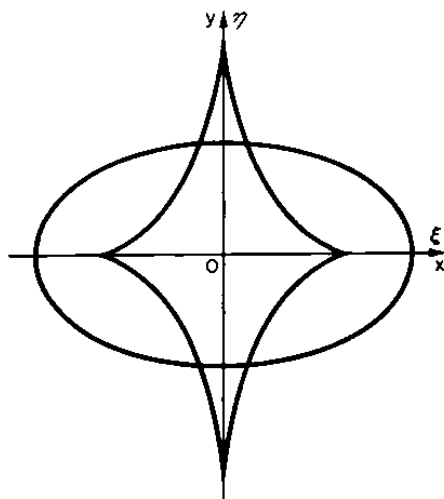


FIG. 107

This curve is obtained from the curve $\xi^{2/3} + \eta^{2/3} = \left(\frac{a^2 - b^2}{a}\right)^{2/3}$ by changing all its ordinates in the ratio $a : b$. The curve $x^{2/3} + y^{2/3} = k^{2/3}$ is called an astroid (Fig. 106) and is a particular case of a hypocycloid—the curve described by any point of a circle rolling without slip on the inside of another circle (if the

rolling circle touches the fixed circle on the outside, the resulting curve is called an epicycloid). The astroid is the hypocycloid obtained when the radius of the rolling circle is a quarter the radius of the fixed circle.

Thus the evolute of an ellipse is a deformed astroid (Fig. 107). The cusps of the astroid correspond to the vertices of the ellipse, the radius of curvature of the ellipse being a minimum ($= b^2/a$) at the vertices on the major axis, and a maximum ($= a^2/b$) at the vertices on the minor axis.

III. EVOLUTE OF CYCLOID. If we take the cycloid

$$x = a(t - \sin t), \quad y = a(1 - \cos t),$$

we get by formulae (B) of Sec. 82:

$$\xi = a(t + \sin t), \quad \eta = a(-1 + \cos t).$$

We introduce the new parameter $\tau = t - \pi$; the equation of the evolute can now be written as

$$\xi = a(\pi + \tau - \sin \tau), \quad \eta = a(-1 - \cos \tau)$$

or

$$\xi' = \xi - a\pi = a(\tau - \sin \tau), \quad \eta' = \eta + 2a = a(1 - \cos \tau).$$

We see from this that the evolute of a cycloid is the cycloid itself, displaced $a\pi$ units to the right along the axis of abscissae and $2a$ units downwards along the axis of ordinates (Fig. 108).

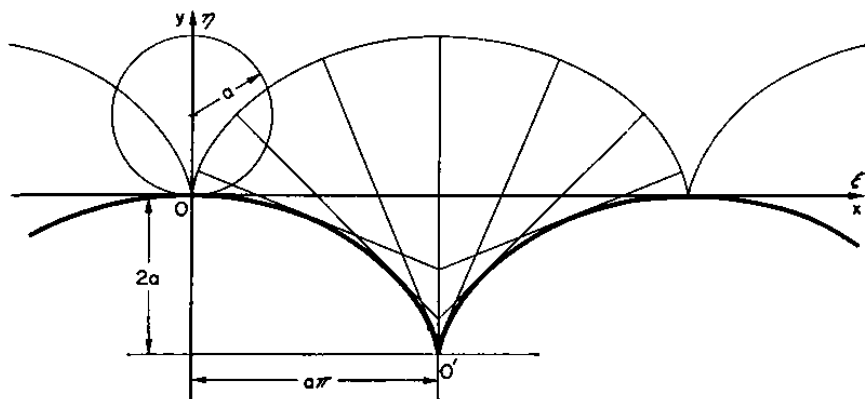


FIG. 108

The cusps of the cycloid-evolute correspond to the vertices of the cycloid-involute. At these vertices the radius of curvature is a maximum, equal to $4a$.

We now notice that the semi-arch of the cycloid-evolute corresponds to the same semi-arch of the cycloid-involute, the difference in the radii of curvature at its ends being equal to $4a$. We can thus conclude (by the second property, Sec. 82) that the length of one arch of the cycloid is equal to $8a$, i.e. is eight times the radius of the generating circle.

THE DEFINITE INTEGRAL

1. The Definite Integral

84. Area of a curvilinear trapezoid. We shall start with the geometrical problem of finding the area of a plane figure. Both geometry and mathematical analysis owe their origins to this problem. The method of solving it has a high degree of generality: it can also be used for finding other quantities, completely different from areas as regards their physical significance.

The theory of areas starts from two propositions:

(1) *the area of a figure made up of several figures is equal to the sum of the areas of these figures;*

(2) *the area of a rectangle is equal to the product of its sides.*

We find in elementary geometry, on the basis of these propositions, the area of a triangle, then the area of a polygon (since every polygon can be split up into triangles). With the aid of some operation of passage to the limit, the area of any figure can be established from the areas of polygons.

It may be recalled that this is precisely the method used in elementary geometry for finding the area of a circle, from the areas of the inscribed and exscribed polygons. But use is made here of special geometrical properties of the figure (the circle) and the process becomes very difficult in other cases.

We shall assume that the figure whose area we want to find is in a plane furnished with a system of Cartesian co-ordinates (for the case of a polar system, see Sec. 117).

Definition. A curvilinear trapezoid is a figure bounded by Ox , by a curve which is cut in not more than one point by any straight line

* The exposition is arranged in Chapters V ("The definite integral") and VI ("The indefinite integral") in such a way that they can be studied in any order: first Chapter V, then VI, or alternatively, first Chapter VI, then V.

parallel to Oy , and by two straight lines $x = a$ and $x = b$ (Fig. 109); the interval $[a, b]$ of Ox is called the base* of the curvilinear trapezoid**.

As a rule, any figure can be split up into a series of curvilinear trapezoids, and hence the required area can be defined as the algebraic sum of the areas of the component curvilinear trapezoids. For instance, the area of the figure illustrated in Fig. 110 can be represented by the following algebraic sum:

$$\text{area } AA'C'C - \text{area } BB'C'C + \text{area } BB'D'D - \text{area } AA'D'D.$$

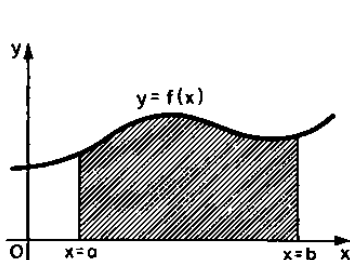


FIG. 109

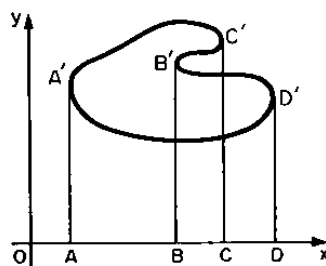


FIG. 110

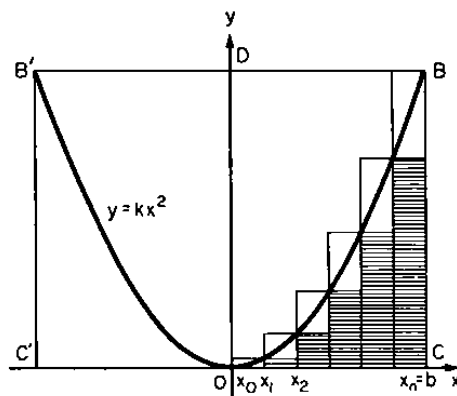


FIG. 111

It follows from this that all we need for the solution of our problem is to find the area of a curvilinear trapezoid.

I. Before passing to the general case, we take for simplicity the parabolic trapezoid (triangle), bounded by the parabola $y = kx^2$, Ox and the straight lines $x = 0$, $x = b$ (Fig. 111).

* This terminology differs from that used in elementary geometry, where the interval $[a, b]$ is called the height of the trapezoid, and the segments of the parallel straight lines $x = a$, $x = b$ are called its bases.

** Obviously, the base of a curvilinear trapezoid can also be situated on Oy .

We construct a step-line close to the parabola as follows: we divide the base of the trapezoid into n equal parts and erect perpendiculars to the base from the points of subdivision as far as their intersections with the parabola; from each point of intersection we draw a straight line parallel to the axis of abscissae as far as its intersection with the following perpendicular.

The area of the step figure thus obtained is easily found.

Let $x_0, x_1, x_2, \dots, x_{n-1}, x_n$ denote the abscissae of the points of subdivision. Since the length of the base is b , we have

$$x_0 = 0, \quad x_1 = \frac{b}{n}, \quad x_2 = 2 \frac{b}{n}, \dots, x_{n-1} = (n-1) \frac{b}{n}, \quad x_n = n \frac{b}{n} = b;$$

the abscissae of the points of subdivision form an arithmetic progression with difference b/n .

Let $y_0, y_1, y_2, \dots, y_{n-1}, y_n$ denote the ordinates corresponding to the points of subdivision, and let s_n be the area.

We obviously have:

$$s_n = y_0(x_1 - x_0) + y_1(x_2 - x_1) + \dots + y_{n-1}(x_n - x_{n-1}).$$

Since the ordinate y_i , corresponding to the point of subdivision x_i is equal to kx_i^2 , whilst $x_i = ib/n$, we have

$$y_i = k i \frac{b^2}{n^2} \quad (i = 0, 1, 2, \dots, n-1).$$

On substituting these values in the expression for s_n , we get:

$$s_n = k \cdot 1^2 \frac{b^2}{n^2} (x_2 - x_1) + k \cdot 2^2 \frac{b^2}{n^2} (x_3 - x_2) + \dots \\ \dots + k \cdot (n-1)^2 \frac{b^2}{n^2} (x_n - x_{n-1}),$$

or

$$s_n = k \cdot 1^2 \frac{b^2}{n^2} \frac{b}{n} + k \cdot 2^2 \frac{b^2}{n^2} \frac{b}{n} + \dots + k \cdot (n-1)^2 \frac{b^2}{n^2} \frac{b}{n},$$

since $x_{i+1} - x_i = b/n$. On taking the common factors outside, we get

$$s_n = k \frac{b^3}{n^3} [1^2 + 2^2 + \dots + (n-1)^2].$$

The sum in brackets is equal to* $\frac{n(n-1)(2n-1)}{6}$.

* This is easily proved, for example, by induction.

Thus

$$s_n = k \frac{b^3}{n^3} \frac{n(n-1)(2n-1)}{6} = k \frac{b^3}{6} \left(1 - \frac{1}{n}\right) \left(2 - \frac{1}{n}\right).$$

This formula gives the area of the n -step figure which is taken as an approximation (on the low side) to the area of the parabolic trapezoid. As n increases the n -step figure approaches the trapezoid OBC (Fig. 111); therefore the area of the trapezoid—let us denote it by s —is defined as the limit to which the function s_n tends on indefinite increase of the integral argument n . This condition gives:

$$s = \lim_{n \rightarrow \infty} s_n = k \frac{b^3}{6} \cdot 2 = k \frac{b^3}{3}.$$

Here, s_n tends to its limit s whilst constantly increasing. We now construct a step-line as follows to replace the parabola: from each point of subdivision of the parabola we draw a straight line parallel to Ox , to its intersection with the previous, instead of the following, perpendicular. This gives us an n -step figure, the area s'_n of which will be greater than s . We have:

$$\begin{aligned} s'_n &= y_1(x_1 - x_0) + y_2(x_2 - x_1) + \cdots + y_n(x_n - x_{n-1}) \\ &= \frac{b}{n} (kx_1^2 + kx_2^2 + \cdots + kx_n^2) = k \frac{b^3}{n^3} (1^2 + 2^2 + \cdots + n^2) \end{aligned}$$

i.e.

$$s'_n = k \frac{b^3}{n^3} \frac{n(n+1)(2n+1)}{6} = k \frac{b^3}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right).$$

This formula also gives an approximation for the area of the trapezoid (on the high side).

On indefinite increase of n , the step figure also approaches indefinitely the trapezoid OBC , so that we can take the limit of s'_n as $n \rightarrow \infty$ as the area s of the trapezoid. But

$$\lim_{n \rightarrow \infty} s'_n = k \frac{b^3}{6} \cdot 2 = k \frac{b^3}{3},$$

and, as would be expected, we get the same value as above. Here, s'_n tends to its limit s whilst constantly decreasing.

We observe that, since $BC = kb^2$, the area of the rectangle $CBB'C'$ is equal to $2bkb^2 = 2kb^3$; hence it follows that the area of the parabolic segment BOB' is equal to

$$2kb^3 - 2k \frac{b^3}{3} = \frac{4}{3} kb^3,$$

i.e. it amounts to two thirds the area of rectangle $CBB'C'$, having as its sides the chord BB' and the "height" OD . This result was obtained by Archimedes by anticipating methods invented many centuries later, which then led to the integral calculus.

If the base of the parabolic trapezoid is the interval $[a, b]$, the area of the trapezoid s becomes

$$s = c \frac{b^3 - a^3}{3}.$$

For, the area of this trapezoid is evidently equal to the difference between the areas of the trapezoids with bases $[0, b]$ and $[0, a]$; by what has been said above, the latter are equal respectively to $kb^3/3$ and $ka^3/3$.

We recommend the reader to use the method just described for finding the area (which will be familiar to him) of a rectilinear trapezoid, formed by the straight lines $y = kx$, $y = 0$, $x = a$, $x = b$.

The area s obtained in this case is

$$s = k \frac{b^2 - a^2}{2},$$

which is also easily found by elementary methods.

Finally, we note that the area s of the rectangle bounded by $y = k$, $y = 0$, $x = a$, $x = b$, is equal to

$$s = k(b - a).$$

II. We turn to the general case, when the curvilinear trapezoid with base $[a, b]$ is bounded by a curve given by the equation $y = f(x)$, where $f(x)$ is a continuous function in the interval $[a, b]$.

We shall assume for the time being that $f(x) > 0$ in $[a, b]$, i.e. that the trapezoid is located above Ox . There is nothing about the method which we used for finding the area of a parabolic trapezoid to prevent us from applying it in the general case. The method consists in subdividing the base of the trapezoid: perpendiculars to the base are erected from the points of subdivision to their intersections with the curve, whilst from each point of the curve thus obtained a straight line is drawn parallel to the base, to its intersection with the following (or the previous—it makes no difference) perpendicular. The area of the step figure thus formed is regarded as an approximation to the area of the curvilinear trapezoid, the approximation being the more accurate, the more closely the step-line replacing the curve borders on the latter. This latter effect is achieved by

increasing the number of points of subdivision of the base whilst simultaneously decreasing the lengths of all the subdivisions.

Definition. The area of a curvilinear trapezoid is the limit to which the variable area of the step-figure tends on indefinite increase of the number of points of subdivision of the base and as the lengths of all the subdivisions tend to zero.

Let the points $x_0 = a, x_1, x_2, \dots, x_{n-1}, x_n = b$ subdivide the interval $[a, b]$ into n parts $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$. These intervals will be termed sub-intervals. Let $y_0, y_1, y_2, \dots, y_{n-1}, y_n$ denote the ordinates of the curve $y = f(x)$ corresponding to the points of subdivision. As above, we construct an n -step figure (Fig. 112).

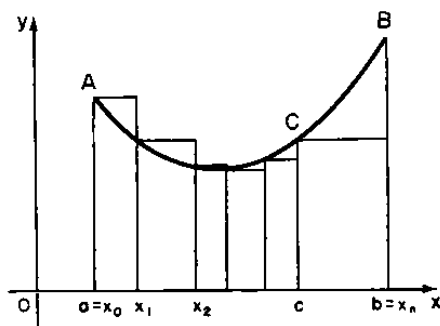


FIG. 112

We previously divided the base into equal parts; but this is not essential; it is only necessary that, on indefinite increase of the number of parts, the length of the greatest of them should tend to zero. If this proviso is not satisfied, it may happen that the step figure will not tend to the curvilinear trapezoid. Suppose, in fact, we increase the number of points of subdivision in such a way that one sub-interval, say (c, b) (Fig. 112) remains invariable; then the step-line can approach indefinitely the arc AC of the curve $y = f(x)$, but will never approach the arc CB , and the constant part $cCBb$ of the trapezoid will never be covered by our figures. We shall therefore not be able to find the area of the trapezoid from the areas of these figures.

Let s_n denote the area of the constructed n -step figure, and s the area of the curvilinear trapezoid which we want to find. We write down an expression for s_n . The total figure consists of n rectangles, the area of each of which is easily found; in fact, the area of the rectangle corresponding to the $(i + 1)$ -th sub-interval is obviously

equal to $y_i(x_{i+1} - x_i)$. Thus

$$s_n = y_0(x_1 - x_0) + y_1(x_2 - x_1) + \cdots + y_i(x_{i+1} - x_i) + \cdots \\ \cdots + y_{n-1}(x_n - x_{n-1}),$$

or, since $y_i = f(x_i)$,

$$s_n = f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + \cdots + f(x_i)(x_{i+1} - x_i) + \cdots \\ \cdots + f(x_{n-1})(x_n - x_{n-1}).$$

All the terms of this sum have the same form: they differ from each other only in the values of the index of the independent variable. We introduce for brevity the symbol Σ (the Greek capital letter "sigma") to denote the sum; thus we have

$$s_n = \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i). \quad (*)$$

This symbol means in general that the given expression has to be summed with the index taking all integral values from the one indicated below the symbol "sigma" to the one indicated above.

We note that each term $f(x_i)(x_{i+1} - x_i)$ of sum (*) can be imagined to be made up as follows: in the sub-interval $[x_i, x_{i+1}]$, in which the function has in general a variation (no matter how small), we take the function as equal to its value at the initial point of the sub-interval, $y = f(x_i)$, then work out the area of the figure thus formed (a rectangle).

The area of the given curvilinear trapezoid is defined as the limit of s_n as the length of the greatest sub-interval tends to zero. Thus,

$$s = \lim_{\max(x_{i+1} - x_i) \rightarrow 0} s_n = \lim_{\max(x_{i+1} - x_i) \rightarrow 0} \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i).$$

85. Examples from physics. We consider some important physical concepts, the strict definitions of which are based on the same arguments as the definition of the area of a curvilinear trapezoid.

I. WORK DONE BY A VARIABLE FORCE. Let a body move along a straight line under the action of a force, directed along the direction of motion. We want to find the work done when the body is displaced from the position M to the position N (Fig. 113).

If the force remains constant throughout the path from M to N , the work is well known to be given by the product of the force and the length of path. Let A denote the work, P the force, and S the

length of path MN . Then*

$$A = PS.$$

We now suppose that the force varies continuously along the path from M to N . At every point between M and N , situated at a distance s from M , the force acting takes a definite value P . This means that the force P is a function of the distance s , i.e. $P = f(s)$. How shall we define in this case the work done when the body is displaced from the point M to the point N ?

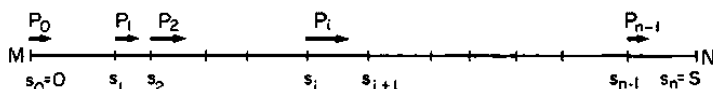


FIG. 113

We divide the entire path MN , i.e. the interval of variation of s , into n parts by points situated at distances $s_0 = 0, s_1, s_2, \dots, s_i, s_{i+1}, \dots, s_{n-1}, s_n = S$ from the point M . Instead of the force P acting along the path MN , we take another force P'_n which preserves a constant value in each of our sub-intervals, these values being set equal to the values of the operative force P say at the initial points. The force will be equal to $P_0 = f(s_0)$ in the first sub-interval $[s_0, s_1]$, equal to $P_1 = f(s_1)$ in the second $[s_1, s_2], \dots$, equal to $P_i = f(s_i)$ in the $(i+1)$ -th $[s_i, s_{i+1}]$, and so on.

We take the work done over the path to be equal to the sum of the works corresponding to the individual parts into which the whole path is divided; hence we obviously have for the work done by the force P'_n :

$$A_n = f(s_0)(s_1 - s_0) + f(s_1)(s_2 - s_1) + \dots + f(s_i)(s_{i+1} - s_i) + \dots \\ \dots + f(s_{n-1})(s_n - s_{n-1}) = \sum_{i=0}^{n-1} f(s_i)(s_{i+1} - s_i).$$

We take A_n to be an approximation to the required work, the more accurate, the greater the number n and the smaller the pieces into which the path is sub-divided. At the same time, the auxiliary force P'_n replacing the true force P differs less and less from P .

* If P is expressed in kilogrammes and S in metres, A will be expressed in kilogramme-metres. Whereas if P is expressed in dynes and S in centimetres, A will be expressed in ergs.

On continuing the argument as in the problem of an area, the number n increases indefinitely whilst the greatest of the sub-intervals tends to zero. Our auxiliary force now tends indefinitely to the given force P . The work A is defined as the limit of A_n as the length of the greatest sub-interval tends to zero:

$$A = \lim_{\max(s_{i+1} - s_i) \rightarrow 0} A_n = \lim_{\max(s_{i+1} - s_i) \rightarrow 0} \sum_{i=0}^{n-1} f(s_i) (s_{i+1} - s_i).$$

II. PATH. We suppose that the body undergoes a gradual motion, the velocity v being known at any instant t in some interval $[T_1, T_2]$. In other words, the velocity v is given as a (continuous) function of time t , i.e. $v = f(t)$. Let us find the path s through which the body moves from time $t = T_1$ to $t = T_2$. It will be seen that this problem is the converse of that considered in Sec. 42, of finding the velocity when the path is given as a function of time.

If the velocity v remains constant throughout the interval $[T_1, T_2]$, i.e. if the motion is uniform, the path s is measured by the product of the velocity and the time during which the body moves: $s = v(T_2 - T_1)$. But if the velocity changes in the course of time, to find the path traversed we must have recourse to an argument analogous to that used on two occasions above for finding area and work. We divide the time interval $[T_1, T_2]$ into n sub-intervals by points $t_0 = T_1, t_1, t_2, \dots, t_i, t_{i+1}, \dots, t_{n-1}, t_n = T_2$. We replace the given motion by another, which is uniform in each sub-interval of time, the velocity v in the interval $[t_i, t_{i+1}]$ being equal to the velocity of the given motion say at the initial instant: $v = f(t_i)$. The path traversed in the time from $t = t_i$ to $t = t_{i+1}$ is now equal to $f(t_i)(t_{i+1} - t_i)$. It is clear that the path s_n , corresponding to the interval $[T_1, T_2]$ is equal to the sum of the distances traversed in the sub-intervals into which $[T_1, T_2]$ is divided. Thus

$$s_n = f(t_0)(t_1 - t_0) + f(t_1)(t_2 - t_1) + \dots + f(t_i)(t_{i+1} - t_i) + \dots \\ \dots + f(t_{n-1})(t_n - t_{n-1}) = \sum_{i=0}^{n-1} f(t_i)(t_{i+1} - t_i).$$

The magnitude s_n is an approximation to the path s , the more accurate, the greater the number n and the smaller the sub-intervals $[t_i, t_{i+1}]$. The limit of s_n as the greatest sub-interval tends to zero in fact gives the required path s :

$$s = \lim_{\max(t_{i+1} - t_i) \rightarrow 0} s_n = \lim_{\max(t_{i+1} - t_i) \rightarrow 0} \sum_{i=0}^{n-1} f(t_i)(t_{i+1} - t_i).$$

We note that the path thus calculated is in fact the same as the path from which we started when we defined velocity in Sec. 42. This will be proved at the end of the chapter (Sec. 93).

III. MASS. We take a segment of a material curve and suppose that we know the density at every point of it. This means that the density δ is given as a (continuous) function of the length s of the curve from one end to the point in question: $\delta = \varphi(s)$. Let us find the mass of the complete curve of length S , i.e. in the interval from $s = 0$ to $s = S$. This is the converse of the problem considered in Sec. 42.

If the density δ is constant throughout the curve, i.e. if the mass is distributed uniformly, the mass m is given by the product of the density and the length of arc: $m = \delta S$. But if the density varies along the curve, we proceed as above: we divide the entire arc into n parts with the aid of points at distances $s_0 = 0, s_1, s_2, \dots, s_i, s_{i+1}, \dots, s_{n-1}, s_n = S$ from the initial point, and suppose that the mass is distributed uniformly in each sub-interval $[s_i, s_{i+1}]$, i.e. that the density remains constant, equal to the given density say at the initial point of the sub-interval: $\delta = \varphi(s_i)$. With this assumption, the mass corresponding to the sub-interval $[s_i, s_{i+1}]$ will be equal to $\varphi(s_i)(s_{i+1} - s_i)$, whilst the mass corresponding to the entire interval $[0, S]$ will obviously be

$$m_n = \varphi(s_0)(s_1 - s_0) + \varphi(s_1)(s_2 - s_1) + \\ + \dots + \varphi(s_i)(s_{i+1} - s_i) + \dots + \varphi(s_{n-1})(s_n - s_{n-1}).$$

As the length of the greatest sub-interval tends to zero (i.e. on indefinite increase also of the number n), the "piecewise uniform" mass distribution will tend indefinitely to the given distribution; consequently, just as in the cases previously discussed, the required mass m is equal to the limit of m_n as the greatest sub-interval tends to zero:

$$m = \lim_{\max(s_{i+1} - s_i) \rightarrow 0} m_n = \lim_{\max(s_{i+1} - s_i) \rightarrow 0} \sum_{i=0}^{n-1} \varphi(s_i)(s_{i+1} - s_i).$$

We shall prove later (Sec. 93) that the mass thus calculated is the same as the mass from which we started when defining density in Sec. 42.

36. The definite integral. Existence theorem. We have seen in the two previous sections that the definition of certain important concepts of geometry and physics (the concepts of area, work, path, mass)

lead to the same sequence of operations on the known functions and their arguments (these operations include the essential one of passage to the limit). Inasmuch as this sequence of operations is applied in various cases and is of great importance, we need to establish it with mathematical strictness, independently of the concrete conditions of the problem. The use of it in any suitable individual case will then be familiar and will not require special arguments, which are sometimes more difficult with particular than with general hypotheses.

If we disregard the physical meaning of the variables and their notation, the sequence of operations consists in the following:

(1) the interval $[a, b]$ in which the continuous function $f(x)$ is given (we now dispense with the assumption that $f(x) > 0$ in the interval) is sub-divided into n sub-intervals with the aid of the points $x_0 = a, x_1, x_2, \dots, x_{n-1}, x_n = b$, where $a < x_1 < x_2 < \dots < x_{n-1} < b$;

(2) the value of the function $f(x_i)$ at the initial point of each sub-interval is multiplied by the length of the sub-interval $x_{i+1} - x_i$, i.e. we form the product $f(x_i)(x_{i+1} - x_i)$;

(3) we take the sum I_n of all these products:

$$\begin{aligned} I_n &= f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + \dots + f(x_{n-1})(x_n - x_{n-1}) \\ &= \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i), \end{aligned}$$

or, if we write $x_{i+1} - x_i = \Delta x_i$:

$$I_n = \sum_{i=0}^{n-1} f(x_i) \Delta x_i; \quad (\text{A})$$

(4) we find the limit I of the sum I_n , as the greatest sub-interval tends to zero:

$$I = \lim_{\max \Delta x_i \rightarrow 0} I_n.$$

In the four concrete cases discussed above, this limit I measures respectively: area, work, path, mass. In the general case it is called the *definite integral*, or simply the *integral* of the function $f(x)$ between the limits a and b , and is written:

$$I = \int_a^b f(x) dx$$

(read as: "the integral of $f(x) dx$ from a to b "). Thus by definition:

$$\int_a^b f(x) dx = \lim_{\max \Delta x_i \rightarrow 0} \sum_{i=0}^{n-1} f(x_i) \Delta x_i. \quad (\dagger)$$

The sum (A) is called the n -th integral sum.

Definition. The **definite integral** ($\dagger$) is the limit to which the n -th integral sum (A) tends when the length of the greatest sub-interval tends to zero.

On applying this new fundamental concept of mathematical analysis to the four concrete problems dealt with in Secs. 84 and 85, the conclusions obtained can be expressed in the following words:

(1) *the area of a curvilinear trapezoid is equal to the integral of the ordinate of the curve bounding the trapezoid over its base:*

$$s = \int_a^b f(x) dx;$$

(2) *the work done by a force is equal to the integral of the force over the path:*

$$A = \int_0^s f(s) ds;$$

(3) *the path traversed by a body is equal to the integral of the velocity over time:*

$$s = \int_{T_1}^{T_2} f(t) dt;$$

(4) *the mass distributed on a curve is equal to the integral of the density over the length of the curve:*

$$m = \int_0^s \varphi(s) ds.$$

Three important remarks must be made in connection with our definition of the integral:

I. Our definition would lose the necessary generality if it were suddenly to happen that integral sum ($\dagger$) for a continuous function did not tend to a limit as $n \rightarrow \infty$. For such a function the concept of integral would become meaningless, as would the visual geometrical concept of area for the corresponding curvilinear trapezoid.

The truth that I_n tends to a definite number for any continuous function may obviously be drawn from direct geometrical considerations; but these cannot serve as sole basis for a theoretical discussion. A general theorem holds, however, according to which every function continuous in a finite interval $[a, b]$ has an integral, i.e. I_n tends to a definite number I as the length of the greatest sub-interval tends to zero.

II. I_n expresses the area of an n -step figure, formed from rectangles whose bases are the sub-intervals and heights the ordinates of the curve $y = f(x)$ at the initial points of the sub-intervals. If we assume it to be established that $I_n \rightarrow I$, this very fact necessitates the assumption that the area of the step figure formed from rectangles, whose heights are the ordinates of the curve at the end-points of the sub-intervals, also tends to I . The area of this latter figure is obviously given by

$$\sum_{i=0}^{n-1} f(x_{i+1}) \Delta x_i. \quad (\text{B})$$

Moreover, suppose we take the n -step figure formed from rectangles, each of which has a sub-interval as base, and as height the ordinate of the curve at any point of this sub-interval (Fig. 114).

It must be assumed, as before, that the area of this figure tends to the area of the curvilinear trapezoid ($= I$) independently of which points in the sub-intervals are taken each time as the bases of the heights of the rectangles. We write these points as $\xi_0, \xi_1, \dots, \xi_{n-1}$:

$$a = x_0 \leq \xi_0 \leq x_1 \leq \xi_1 \leq x_2 \leq \dots \leq x_{n-1} \leq \xi_{n-1} \leq x_n = b.$$

The area of the step figure is written as:

$$\sum_{i=0}^{n-1} f(\xi_i) \Delta x_i. \quad (\text{C})$$

Consequently, we must have

$$\lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} f(x_i) \Delta x_i = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} f(x_{i+1}) \Delta x_i = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} f(\xi_i) \Delta x_i.$$

Since sums (A) and (B) are obtained from sum (C) with $\xi_i = x_i$ and $\xi_i = x_{i+1}$ respectively, our conclusion amounts to the fact that sum (C) must have the same limit for any values of ξ_i , $x_i \leq \xi_i \leq x_{i+1}$. This is in fact true.

III. We have not made any special provisos regarding the method of sub-dividing the interval $[a, b]$ into n sub-intervals, i.e. the law by

which the abscissae of the points of division $x_1, x_2, \dots, x_{n-1}$ are selected. It is completely obvious geometrically, that if I_n tends to a definite number I for any given method of sub-dividing the interval, this must be the case for any other method, provided the number of points of sub-division increases indefinitely, whilst the length of the greatest sub-interval at the same time tends to zero.

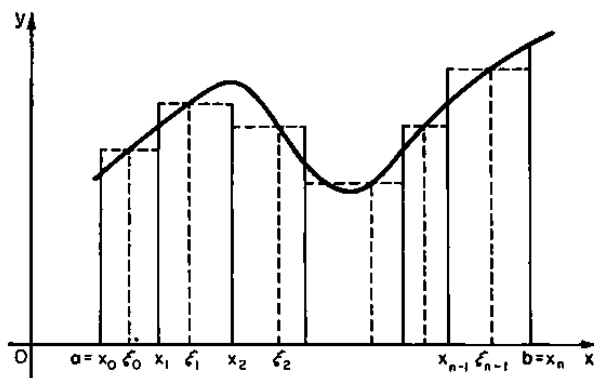


FIG. 114

For, the sum (C) tends to the same limit independently of the law by which the interval of integration is sub-divided.

We can widen our earlier concept of n -th integral sum by calling the sum (C) the " n -th integral sum" for function $f(x)$. The following general theorem holds.

*Existence theorem for the definite integral**. Given a finite interval $[a, b]$ of variation of the variable x , $a \leq x \leq b$, and a function $f(x)$ continuous in it, the corresponding n -th integral sum tends to a limit when the length of the greatest sub-interval tends to zero, the limit of the sum being the same, independently of what values of x are taken in the sub-intervals $[x_i, x_{i+1}]$ as the numbers ξ_i , and independently of the method of dividing the interval $[a, b]$ into sub-intervals $[x_i, x_{i+1}]$.

This limit is called the definite integral or simply the integral of function $f(x)$ between the limits a and b :

$$\lim_{\max \Delta x_i \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_i) \Delta x_i = \int_a^b f(x) dx.$$

* See a more complete course of analysis for a proof of the existence theorem for a definite integral.

The symbol of the integral indicates its origin: $\int$ is an elongated letter *s*, the first letter of the word *summa*; the expression standing behind (we also say: under) the integral sign indicates the form of the summed terms; the index is omitted for the variable in the expression under the integral, which underlines the fact that, in the process of summation, completed by a passage to the limit, the variable x takes all values in the interval $[a, b]$; the numbers below and above the integral sign indicate the ends of the interval over which the summation is taken.

The function $f(x)$ is called the integrand; the expression $f(x) dx$ is the integrand expression; the number a is the lower, and b the upper limit* of the integral; the variable x is the variable of integration; and the interval $a \leq x \leq b$ is the interval of integration.

Integral sums formed with different sub-divisions of the interval of integration and different choices of points ξ can differ very substantially from each other. The remarkable theorem stated above shows that the difference between them disappears as the number of points of sub-division increases and the greatest sub-interval

decreases, and vanishes completely in the limit. The symbol $\int_a^b f(x) dx$ simply denotes a number; it therefore contains no trace of the special method of obtaining the number; it does not depend on how the operation of forming the definite integral is carried out. This number is naturally also independent of the notation for the variable of integration, so that, for example,

$$\int_a^b f(x) dx = \int_a^b f(u) du.$$

37. Evaluation of the definite integral. The operation finding the definite integral for a given integrand and given interval of integration is called definite integration of the function.

Definite integration, carried out in accordance with the definition of a definite integral, represents as a rule an extremely laborious operation. For illustration, let us find the integral of the power function with positive integral index k :

$$I^{(k)} = \int_a^b x^k dx, \quad a < x < b.$$

* The term "limit" is used here in a sense which has no connection with the "limit of a function".

This integral is already known to us in three particular cases, $k = 0$, $k = 1$, $k = 2$ (Sec. 84):

$$I^{(0)} = b - a, \quad I^{(1)} = \frac{b^2 - a^2}{2}, \quad I^{(2)} = \frac{b^3 - a^3}{3}.$$

The regularity is easily discovered here, whereby the idea suggests itself that, for any integral $k > 0$:

$$I^{(k)} = \frac{b^{k+1} - a^{k+1}}{k+1}.$$

We shall confirm this conjecture by direct evaluation.

In the three cases mentioned, we divided the interval $[a, b]$ into n equal parts with the aid of the points $x_1, x_2, \dots$, forming an arithmetic progression. In the case of an arbitrary integer $k > 0$, it is more convenient to divide the interval into n unequal parts, with the aid of the points $x_0 = a, x_1, x_2, \dots, x_{n-1}, x_n = b$ forming a geometric progression. Let the ratio of the progression be q . Then

$$x_0 = a, \quad x_1 = aq, \quad x_2 = aq^2, \dots, x_i = aq^i, \dots, x_n = b = aq^n.$$

We have from the last equation:

$$q = \left(\frac{b}{a}\right)^{\frac{1}{n}}, \quad (*)$$

and since $b > a$, we have $q > 1$. The lengths of the sub-intervals are equal to:

$$x_1 - x_0 = a(q - 1), \quad x_2 - x_1 = aq(q - 1), \dots \\ x_{i+1} - x_i = aq^i(q - 1), \dots, \quad x_n - x_{n-1} = aq^{n-1}(q - 1).$$

As we see, they form another geometric progression with ratio q . Since $q > 1$, each sub-interval is longer than all the previous ones, i.e. the longest is the last one:

$$x_n - x_{n-1} = aq^n \frac{q - 1}{q} = b \frac{q - 1}{q}.$$

Since $q \rightarrow 1$ as $n \rightarrow \infty$ (as follows from formula (*)), the length of the greatest sub-interval tends to zero, so that our present system of sub-division of the interval $[a, b]$ satisfies the necessary conditions.

We form the n -th integral sum:

$$I_n^{(k)} = \sum_{i=0}^{n-1} x_i^k \Delta x_i = \sum_{i=0}^{n-1} a^k q^{ik} \cdot a q^i (q - 1);$$

since $a^{k+1}(q-1)$ appears as a factor in all the terms, we have

$$\begin{aligned} I_n^{(k)} &= a^{k+1}(q-1) \sum_{i=0}^{n-1} q^{i(k+1)} \\ &= a^{k+1}(q-1) [1 + q^{k+1} + q^{2(k+1)} + \dots + q^{(n-1)(k+1)}]. \end{aligned}$$

The square brackets contain the sum of n terms of a geometric progression with ratio q^{k+1} . We get:

$$I_n^{(k)} = a^{k+1}(q-1) \frac{q^{n(k+1)} - 1}{q^{k+1} - 1},$$

which, in view of the relationship

$$q^{k+1} - 1 = (q-1)(q^k + q^{k-1} + \dots + q + 1), \quad q^n = \frac{b}{a},$$

gives

$$\begin{aligned} I_n^{(k)} &= a^{k+1}(q-1) \frac{b^{k+1}/a^{k+1} - 1}{(q-1)(q^k + q^{k-1} + \dots + q + 1)} \\ &= \frac{b^{k+1} - a^{k+1}}{q^k + q^{k-1} + \dots + q + 1}. \end{aligned}$$

To obtain the integral $I^{(k)}$, it remains to find the limit of this expression as $n \rightarrow \infty$. But if $n \rightarrow \infty$, then $q \rightarrow 1$ and

$$\lim_{n \rightarrow \infty} I_n^{(k)} = \lim_{q \rightarrow 1} \frac{b^{k+1} - a^{k+1}}{q^k + q^{k-1} + \dots + q + 1} = \frac{b^{k+1} - a^{k+1}}{k+1},$$

which is what we wanted to show †.

We shall show later that the formula

$$\int_a^b x^k dx = \frac{b^{k+1} - a^{k+1}}{k+1}$$

holds for any k , excepting $k = -1$, for which the right-hand side becomes meaningless.

The above example shows that evaluation of the definite integral with the aid of the method following directly from the definition involves difficulties in what are apparently the simplest cases. This

† As previously for $k = 2$, we could have divided $[a, b]$ into equal parts. The same result would have been obtained, as follows from the existence theorem. Division into non-equal parts by points forming a geometric progression has the advantage here that it is not necessary to find a formula for the sums of k -th powers of successive integers, the formula for the sum of the terms of a geometric progression being familiar.

vital concept for mathematical analysis would not in fact have achieved its present position, had it not been for the discovery of an incomparably easier and more convenient technique for evaluating definite integrals in practice.

The next article describes a series of fundamental properties of the definite integral which eventually lead us to establishing a roundabout way for evaluating integrals.

2. Basic Properties of the Definite Integral

88. Elementary properties of the definite integral.

THEOREM I (ON THE INTEGRAL OF A SUM). The integral of the sum of a finite number of functions is equal to the sum of the integrals of the separate functions:

$$\begin{aligned} & \int_a^b [f(x) + \varphi(x) + \cdots + \psi(x)] dx \\ &= \int_a^b f(x) dx + \int_a^b \varphi(x) dx + \cdots + \int_a^b \psi(x) dx. \end{aligned}$$

THEOREM II (ON TAKING OUTSIDE A CONSTANT FACTOR). A constant factor in the integrand can be taken outside the integral sign:

$$\int_a^b c f(x) dx = c \int_a^b f(x) dx.$$

These theorems may be proved simply by writing the expressions for the integral sums then using the familiar theorems on the limit of a sum and the limit of a product.

THEOREM III (ON THE SIGN OF AN INTEGRAL). If the integrand does not change sign in the interval of integration, the integral is a number of the same sign as the function.

Proof. Let $f(x) \geq 0$ in the interval $[a, b]$, $a < b$. Then all the terms are non-negative in the integral sum

$$I_n = \sum_{i=0}^{n-1} f(\xi_i) \Delta x_i,$$

i.e. $I_n \geq 0$, and the limit of a non-negative quantity cannot be negative. The definite integral of a continuous function of constant sign $f(x)$ can only be equal to zero when $f(x)$ is identically zero (see Sec. 90).

If the integrand changes sign in the interval of integration, the integral may be either positive or negative or zero.

This property of the definite integral should be borne in mind when solving problems. For instance, when finding the area of a trapezoid with the aid of an integral, its disposition with respect to Ox must be taken into account. When the trapezoid lies wholly above Ox , the integral of the ordinate expresses the area; when the trapezoid lies wholly below Ox , the integral, being negative, obviously expresses the area taken with the negative sign; finally, when the trapezoid lies partly above and partly below Ox , the integral of the ordinate by no means expresses the required area.

Therefore, to find the area of a curvilinear trapezoid, we need to evaluate separately the integrals expressing the areas of the parts lying above Ox , and the integrals for the areas of the parts below Ox , then add the absolute values of the results.

89. Change of direction and sub-division of the interval of integration.

Geometrical interpretation of the integral. We have so far assumed that the lower limit of the integral is less than the upper ($a < b$), or, as we can say, that the interval of integration is directed to the right.

We now extend our definition of integral to the case when the interval of integration is directed to the left, i.e. when the lower limit is greater than the upper ($a > b$). In either case, in fact, we put

$$\int_a^b f(x) dx = \lim_{\max |x_{i+1} - x_i| \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_i) (x_{i+1} - x_i); \quad (*)$$

if $a > b$, then

$$a = x_0 > x_1 > x_2 > \cdots > x_{n-1} > x_n = b, \quad x_i \geq \xi_i \geq x_{i+1}.$$

We shall now justify this extension of the concept of definite integral by proving the following theorem.

THEOREM IV (ON INTERCHANGING THE LIMITS OF AN INTEGRAL). If the upper and lower limits of an integral change places, i.e. the direction of the interval of integration is reversed, the integral merely changes sign:

$$\int_a^b f(x) dx = - \int_b^a f(x) dx. \quad (**)$$

Proof. All the increments of the variable of integration $\Delta x_i = x_{i+1} - x_i$ are negative in sum(*) when $a > b$. We take -1

outside the brackets and outside the limit symbol and reverse the sequence of the terms; this gives

$$\int_a^b f(x) dx = - \lim_{\max |\Delta x_i| \rightarrow 0} \sum_{i=0}^{n-1} f(\xi_{n-i-1}) (x_{n-i-1} - x_{n-i}).$$

Here, the summation* is carried out from left to right:

$$b = x_n \leq \xi_{n-1} \leq x_{n-1} \leq \dots \leq x_1 \leq \xi_0 \leq x_0 = a,$$

and by the first definition of integral, the right-hand side is equal to the integral $\int_b^a f(x) dx$, in which the lower limit is less than the upper, taken with the $-$ sign.

The theorem is proved.

It is clear that equation (**) also holds for $a < b$.

We shall assume that

$$\int_a^a f(x) dx = 0,$$

i.e. *an integral with equal limits vanishes*. This is natural, since, if the end of the trapezoid base coincides with the beginning, the trapezoid becomes a straight line—the ordinate $f(a)$, the “area” of which must be reckoned zero. Equation (**) is thus seen to hold for any relationship between a and b ($a > b$, $a < b$, $a = b$).

The first two properties of the integral (Sec. 88) also relate to the case when $a > b$ in $\int_a^b f(x) dx$, whilst the statement of theorem III must be modified in an obvious way if $a > b$.

THEOREM V (ON SUB-DIVIDING THE INTERVAL OF INTEGRATION). If the interval of integration $[a, b]$ is split into two parts $[a, c]$ and $[c, b]$, then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx. \quad (***)$$

Proof. Since the limit of the integral sum does not depend on the method of sub-dividing the interval $[a, b]$, we shall always arrange for the point c to be a point of sub-division. The integral sum can now be written as

$$\sum f(\xi_i) \Delta x_i = \sum_1 f(\xi_i) \Delta x_i + \sum_2 f(\xi_i) \Delta x_i,$$

* It is useful for clarity to change the notation for the points of sub-division x and the intermediate points ξ , by putting $\xi_{n-i-1} = \xi'_i$, $x_{n-i} = x'_i$.

where all the elements corresponding to points of sub-division of the interval $[a, c]$ are collected in the first sum on the right-hand side, and all the elements corresponding to points of sub-division of $[c, b]$ in the second sum. Both the first and second sums are integral sums for $f(x)$, corresponding to the intervals $[a, c]$ and $[c, b]$. If the number of points of subdivision increases indefinitely, whilst the length of the greatest sub-interval tends to zero for the whole of $[a, b]$, the same must evidently be true for intervals $[a, c]$ and $[c, b]$; now, the first sum tends to the integral between the limits a and c , and the second to the integral between the limits c and b , so that we obtain the required equation.

Relationship(***) expresses the geometrically obvious fact: if a trapezoid is split into two parts, one with base $[a, c]$, the other with base $[c, b]$, the area of the whole trapezoid will be equal to the sum of the areas of the two parts.

(***) also holds in the case when c lies outside the interval $[a, b]$, either to the right ($a < b < c$) or to the left ($c < a < b$) provided that $f(x)$ is continuous in $[a, c]$ or in $[c, b]$.

Let $a < b < c$. By what has been proved:

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx,$$

whence

$$\int_a^b f(x) dx = \int_a^c f(x) dx - \int_b^c f(x) dx.$$

On interchanging the limits of the second integral on the right-hand side, we get:

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

This is what we had to prove.

Similarly, it can be shown that (***) holds when $c < a < b$. We leave the geometrical interpretation of the last two cases to the reader.

It follows directly from the last theorem that, if $c_1, c_2, \dots, c_k$ are any numbers distributed in the interval of continuity of $f(x)$, we have

$$\int_a^b f(x) dx = \int_a^{c_1} f(x) dx + \int_{c_1}^{c_2} f(x) dx + \dots + \int_{c_k}^b f(x) dx.$$

We can now pass to the geometrical interpretation of the definite integral. We agree to ascribe a plus sign to the area of a curvilinear trapezoid situated above Ox , and a minus sign when below Ox . The definite integral of $f(x)$ now evidently becomes the sum of the algebraic areas (i.e. furnished with definite signs) of the curvilinear trapezoids from which the given trapezoid is made up.

Bearing this in mind, from now on *we can always interpret the definite integral*

$$\int_a^b f(x) dx$$

as the "area" (algebraic, not geometric) of the curvilinear trapezoid with base $[a, b]$ and bounded by the curve $y = f(x)$, independently of the concrete nature of the variable of integration x and of the function $f(x)$.

90. Estimation of the definite integral.

I. BASIC THEOREM. We shall indicate the bounds between which the value of an integral certainly lies.

THEOREM VI (ON ESTIMATING A DEFINITE INTEGRAL). The value of a definite integral lies between the products of the greatest and least values of the integrand and the length of the interval of integration, i.e.

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a), \quad b > a,$$

where m and M are respectively the least and greatest values of $f(x)$ in the interval $[a, b]$.

Proof. We take two functions $M - f(x)$ and $m - f(x)$. The first is non-negative in the interval $[a, b]$, the second non-positive. Thus, by property III (Sec. 88):

$$\int_a^b [M - f(x)] dx \geq 0 \quad \text{and} \quad \int_a^b [m - f(x)] dx \leq 0,$$

whilst by property I (Sec. 88):

$$\int_a^b M dx - \int_a^b f(x) dx \geq 0 \quad \text{and} \quad \int_a^b m dx - \int_a^b f(x) dx \leq 0,$$

whence

$$M \int_a^b dx = M(b-a) \geq \int_a^b f(x) dx$$

and

$$m \int_a^b dx = m(b-a) \leq \int_a^b f(x) dx.$$

This is what we wanted to prove.

The geometrical illustration is as follows: the area of the curvilinear trapezoid is greater than the area of the rectangle with base equal to the base of the trapezoid and height equal to the least ordinate of the trapezoid, and is less than the area of the rectangle with the same base and height equal to the greatest ordinate of the trapezoid (Fig. 115).

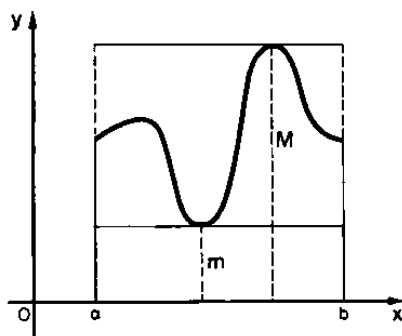


FIG. 115

Having found the bounds for the integral, we can so to speak estimate it. It may happen that it is very difficult, or even impossible to find the exact value of an integral, whilst by estimating it, we recognize, even if crudely, the order of the number by which it is expressed. This type of estimation is encountered fairly often in mathematics.

As a first application of theorem VI, we prove that the integral of a continuous function $f(x)$ of constant sign in the interval of integration $[a, b]$ can only vanish if the integrand is identically zero (see theorem III).

Let us suppose the converse: let the integral vanish, whilst $f(x)$ differs from zero at at least one point inside the interval of integration:

$$I = \int_a^b f(x) dx = 0, \quad \text{whilst} \quad f(c) \neq 0, \quad \text{where} \quad a \leq c \leq b,$$

whilst we suppose $f(x) \geq 0$ in the interval $[a, b]$.

In view of the continuity of $f(x)$ there exists in $[a, b]$ an interval—call it $[\alpha, \beta]$ —containing the point c , in which $f(x)$ differs from zero; let $m \neq 0$ denote the least value of $f(x)$ in $[\alpha, \beta]$. We have by theorem V:

$$\int_a^b f(x) dx = \int_a^\alpha f(x) dx + \int_\alpha^\beta f(x) dx + \int_\beta^b f(x) dx,$$

and since the first and third integrals on the right-hand side are non-negative (theorem III), we obtain by using theorem VI:

$$I = \int_a^b f(x) dx \geq \int_a^\beta f(x) dx > m(\beta - \alpha) \neq 0.$$

This contradicts the condition that $I = 0$; hence $f(x) \equiv 0$.

The bounds of the integral indicated in theorem VI are the more accurate (the narrower), the shorter the interval of integration, and the less the curve $y = f(x)$ differs from the position of a straight line parallel to Ox . The theorem on the estimation of a definite integral enables us to indicate its approximate value.

Examples. (1) Let us estimate the integral

$$\int_0^2 \frac{5-x}{9-x^2} dx.$$

We find by the familiar methods of the differential calculus that the greatest and least values of the integrand in the interval $[0, 2]$ are $3/5$ and $1/2$ respectively. Thus

$$\frac{1}{2} (2 - 0) \leq \int_0^2 \frac{5-x}{9-x^2} dx \leq \frac{3}{5} (2 - 0),$$

i.e. the integral lies between 1 and 1.2. We can reckon as a first approximation that it is equal to 1.1, where the absolute error does not exceed $1.2 - 1 = 0.2$, and the relative does not exceed $\frac{0.2 \cdot 100}{1.1} \approx 18\%$. (We shall later be able to find the exact value of the integral. It is equal to $3/4 \ln 5 - \ln 3 \approx 1.0472$.)

(2) Let us estimate

$$\int_{\pi/4}^{\pi/2} \frac{\sin x}{x} dx.$$

The reader will easily show that the integrand is decreasing in the interval $[\pi/4, \pi/2]$, so that

$$\frac{1}{2} = \frac{\sin \frac{\pi}{2}}{\frac{\pi}{2}} \cdot \frac{\pi}{4} \leq \int_{\pi/4}^{\pi/2} \frac{\sin x}{x} dx \leq \frac{\sin \frac{\pi}{4}}{\frac{\pi}{4}} \cdot \frac{\pi}{4} = \frac{1}{2} \sqrt{2}.$$

The integral therefore lies between 0.5 and 0.71, which gives us not too bad an idea of it. More accurate methods show that it is approximately equal to 0.62.

II. GENERALIZATION OF THE BASIC THEOREM. (1) *Integration of inequalities*. We have the following theorem, more general than theorem VI.

THEOREM. If

$$\psi(x) \leq f(x) \leq \varphi(x)$$

at every point x of an interval $[a, b]$, we have

$$\int_a^b \psi(x) dx \leq \int_a^b f(x) dx \leq \int_a^b \varphi(x) dx, \quad b > a.$$

This means that an inequality between functions involves an inequality in the same sense between their definite integrals, or more briefly, that inequalities can be integrated. It may easily be seen from simple geometrical considerations that differentiation of inequalities is in general impossible.

We shall not dwell on the proof and geometrical interpretation of this theorem since they are precisely similar to the above.

In the particular case when $\varphi(x)$ is identically equal to M , and $\psi(x)$ identically equal to m , we get theorem VI.

(2) *Estimation of the absolute value of a definite integral*. Suppose that, in the interval $[a, b]$:

$$|f(x)| \leq M, \quad \text{i.e.} \quad -M \leq f(x) \leq M.$$

On the basis of property VI:

$$-M(b-a) \leq \int_a^b f(x) dx \leq M(b-a), \quad b > a,$$

i.e.

$$\left| \int_a^b f(x) dx \right| \leq M(b-a).$$

This gives an estimate for the absolute value of a definite integral in accordance with an estimate of the absolute value of the integrand.

III. BUNYAKOVSKII'S INEQUALITY*. If the integrand is written as the product of two functions: $f(x) = f_1(x)f_2(x)$, we have

$$\left| \int_a^b f_1(x)f_2(x) dx \right| \leq \sqrt{\int_a^b [f_1(x)]^2 dx} \cdot \sqrt{\int_a^b [f_2(x)]^2 dx}.$$

Proof. We take the auxiliary integral

$$I = \int_a^b [\lambda f_1(x) + f_2(x)]^2 dx.$$

Given any real λ , this integral cannot be negative (theorem III): $I \geq 0$. On removing the brackets in the integrand and writing

$$\int_a^b [f_1(x)]^2 dx = I_1, \quad \int_a^b f_1(x)f_2(x) dx = I_{12}, \quad \int_a^b [f_2(x)]^2 dx = I_2,$$

we get

$$I_1 \lambda^2 + 2I_{12} \lambda + I_2 \geq 0.$$

The left-hand side of this relationship is a quadratic expression in λ ; it is non-negative for any λ , which is only the case (in view of the fact that $I_1 > 0$) when the discriminant of the expression is non-positive (see Sec. 18):

$$I_{12}^2 - I_1 \cdot I_2 \leq 0,$$

i.e.

$$\left(\int_a^b f_1(x)f_2(x) dx \right)^2 \leq \int_a^b [f_1(x)]^2 dx \cdot \int_a^b [f_2(x)]^2 dx,$$

whence

$$\left| \int_a^b f_1(x)f_2(x) dx \right| \leq \sqrt{\int_a^b [f_1(x)]^2 dx} \cdot \sqrt{\int_a^b [f_2(x)]^2 dx}.$$

Example. Let,

$$I = \int_0^1 \sqrt{1+x^3} dx.$$

We have by theorem VI:

$$1 \leq I \leq \sqrt{2},$$

* V. Ya. Bunyakovskii (1804–1889), academician and famous Russian mathematician, published this inequality in 1859. It is usually (but wrongly!) called “Schwartz’s inequality” in the literature.

whilst we find by Bunyakovskii's inequality, taking $f_1(x) = 1$, $f_2(x) = \sqrt{1+x^3}$:

$$1 \leq \sqrt{1 \cdot \int_0^1 (1+x^3) dx} = \frac{1}{2} \sqrt{5} = \sqrt{\frac{5}{8}} \cdot \sqrt{2},$$

i.e. a better estimate than by theorem VI.

3. Basic Properties of the Definite Integral (Continued).

The Newton-Leibniz Formula

91. Mean value theorem. Mean value of a function.

I. MEAN VALUE THEOREM. The definite integral has the following important property:

THEOREM VII (MEAN VALUE THEOREM). If a function $f(x)$ is continuous in the interval $[a, b]$, there exists at least one value $x = \xi$ in the interval $a \leq \xi \leq b$, for which

$$\frac{\int_a^b f(x) dx}{b-a} = f(\xi). \quad (\dagger)$$

Proof. We have by theorem VI:

$$m \leq \frac{\int_a^b f(x) dx}{b-a} \leq M$$

i.e.

$$\frac{\int_a^b f(x) dx}{b-a} = \mu,$$

where μ is a number lying between the least (m) and the greatest (M) value of $f(x)$ in the interval $[a, b]$, i.e. $m \leq \mu \leq M$. But $f(x)$, being a continuous function, must take at least once every value lying between m and M . Consequently, $f(x)$ takes the value μ for some $x = \xi$, $a \leq \xi \leq b$, i.e. $f(\xi) = \mu$.

This is what we had to prove.

We find from equation(†):

$$\int_a^b f(x) dx = f(\xi) (b - a), \quad a \leq \xi \leq b.$$

This formula enables the mean value theorem to be stated as follows:

The definite integral of a continuous function is equal to the product of the value of the function at some "mean" point of the interval of integration and the length of the interval.

Geometrically the mean value theorem can be interpreted as: there exists a rectangle, with base equal to the trapezoid base, and height equal to the ordinate of the curve bounding the trapezoid at some point of the base, the area of which is equal to the area of the trapezoid (Fig. 116).

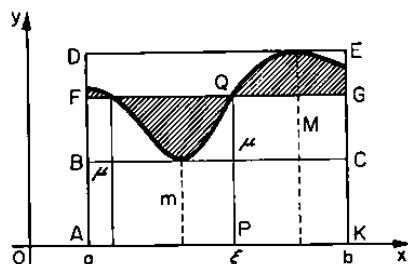


FIG. 116

The theorem may be explained visually. When a straight line moves parallel to Ox (Fig. 116), upwards from the position BC , the area of rectangle $ABCK$ will increase continuously from a value less than to a value greater than the area of the trapezoid. Obviously, at some intermediate position of the line—call it FG —the area of rectangle $AFGK$ will be precisely equal to the trapezoid area s . Since the line always cuts the curve during this motion, in the position FG there will be one (or several: two in Fig. 116) point of intersection Q , whose abscissa will in fact be the "mean" value ξ required by the theorem.

Only in one case, viz. when $f(x)$ is a linear function, i.e. the trapezoid is bounded by a straight line, can we say directly what the point ξ is: it lies at the mid-point of the base (prove this analytically!): $\xi = \frac{1}{2}(a + b)$. The segment PQ will now be the centre line of the rectilinear trapezoid. Sometimes PQ is also called in the general case the "centre line" of the curvilinear trapezoid. A curvilinear trapezoid can have several "centre lines" (equal in size).

The mean value theorem has great theoretical interest, but is of little practical use for evaluating integrals, since we do not know in advance the value of ξ .

II. THE CONCEPT OF ARITHMETIC MEAN. The value ξ found by the mean value theorem is called the mean or, more precisely, the arithmetic mean value of function $f(x)$ in the interval $[a, b]$.

Definition. The arithmetic mean (mean) value y_{av} of a function $y = f(x)$ continuous in an interval $[a, b]$ is the ratio of the definite integral of the function to the length of the interval:

$$y_{av} = \frac{1}{b-a} \int_a^b f(x) dx.$$

We shall make a few remarks on the basis of this definition.

Let a magnitude y take n values: $y_1, y_2, \dots, y_n$. The fraction $\frac{y_1 + y_2 + \dots + y_n}{n}$ is called the mean, or more precisely, the "arithmetic mean" value of the magnitude. For instance, if the atmospheric temperature is measured every hour during a day and night, the "mean" will be the sum of all the observed temperatures divided by 24.

Instead of considering an entire set of values, we can often confine ourselves to indicating only the mean value, since it in a sense characterizes the set. If we say that the mean atmospheric temperature during a day and night is equal to 10.5° , this gives an idea of the temperature regime as a whole, although the temperature at individual hours may be substantially different from 10.5° , either above or below.

But now suppose that the magnitude is measured continuously (say the atmospheric temperature is known at any instant of the 24 hours), and that we want to characterize the entire set of values "on the average". How is the "mean" air temperature to be defined in this case, taking into account the entire set of temperature figures? In general, what should be taken as the "mean value" of a continuous function $y = f(x)$ in an interval $[a, b]$?

We divide $[a, b]$ into n equal parts with the aid of the points $x_0 = a, x_1, x_2, \dots, x_{n-1}, x_n = b$ and take the values of the function at these points:

$$y_0 = f(x_0), \quad y_1 = f(x_1), \dots, y_{n-1} = f(x_{n-1}).$$

We neglect for the moment the values of our function at all the

remaining points of the interval. We take the arithmetic mean of our n values:

$$\eta_n = \frac{y_0 + y_1 + \cdots + y_{n-1}}{n}.$$

It is clear that, the greater n , the more values of the function are taken into account in finding the mean value, so that it is natural to take as the "mean" value y_{av} of the function the limit to which η_n tends as $n \rightarrow \infty$. Let us find this limit.

On multiplying and dividing the expression for η by $b - a$, we get:

$$\eta_n = \frac{1}{b-a} \left(y_0 \frac{b-a}{n} + y_1 \frac{b-a}{n} + \cdots + y_{n-1} \frac{b-a}{n} \right);$$

but since $\frac{b-a}{n} = \Delta x_i$, we have

$$\eta_n = \frac{1}{b-a} \sum_{i=0}^{n-1} y_i \Delta x_i = \frac{1}{b-a} \sum_{i=0}^{n-1} f(x_i) \Delta x_i,$$

whence, on passing to the limit, we get the expression indicated above for the mean value:

$$y_{av} = \lim_{n \rightarrow \infty} \frac{1}{b-a} \sum_{i=0}^{n-1} f(x_i) \Delta x_i = \frac{1}{b-a} \int_a^b f(x) dx.$$

Using the mean value theorem (theorem VII), we conclude that $y_{av} = f(\xi)$, where $a < \xi < b$, i.e. that the "mean" value of a continuous function in an interval is always (provided the function is not constant) less than certain of its values and greater than others, and equal to at least one value (at the "mean" point of the interval).

Geometrically, the "mean" value of a function is represented by the "centre line" of the corresponding curvilinear trapezoid.

The concept of mean value of a function is very useful in practice. Many magnitudes are often characterized by their mean values, as for instance steam pressure, current and voltage of alternating current, speed of a chemical reaction etc.

Sometimes the same magnitude can be regarded as a function now of one variable, now of another. Its mean values with respect to these different variables will then be in general different from each other. For instance, for motion according to the law $s = \frac{1}{2}gt^2$ (free fall in space), the velocity v can be written as a function of time: $v = gt$, and as a function of path: $v = \sqrt{2gs}$. We shall find the mean values of the square of the velocity with respect to time and with respect to the path.

The mean value of v^2 during the time from 0 to T is

$$(v^2)_{av} = \frac{1}{T} \int_0^T g^2 t^2 dt = \frac{1}{3} g^2 T^2;$$

whereas the mean value of v^2 during the path from 0 to S ($=\frac{1}{2}gT^2$) is

$$(v^2)_{av} = \frac{1}{S} \int_0^S 2gs ds = gS = \frac{1}{2} g^2 T^2.$$

It should thus be remembered that the concept of mean value is relative, and defined not only by the set of values of the magnitude, but also by the choice of the variable with respect to which the mean value is considered.

92. Derivative of the integral with respect to its upper limit. We shall regard the lower limit of the integral as constant, and the upper limit as variable. On assigning different values to the latter, we get corresponding values of the integral; hence the integral becomes a function of its upper limit.

As regards the generally accepted notation, the independent variable in the upper limit is usually denoted by the same letter (say x) as denotes the variable of integration. For instance, we write

$$I(x) = \int_a^x f(x) dx.$$

But the variable x in the integrand expression has no relationship to the argument of function $I(x)$; it merely serves as an auxiliary variable—the variable of integration, which runs through values from a to x during the process of forming the integral (summation)—the last x being the upper limit. If we want to evaluate a particular value of the function $I(x)$, say with $x = b$, i.e. $I(b)$, we substitute b for x in the upper limit but naturally do not substitute b for the variable of integration. It would therefore give a clearer picture if we wrote

$$I(x) = \int_a^x f(t) dt,$$

taking another letter (t in this case) for the variable of integration. However, since this variable varies along the same axis, we shall often denote by the same letter the variable of integration and the independent variable in the upper limit, whilst always remembering their different meanings in the integral symbol.

The properties of the integral described in the previous article also relate to an integral with variable upper limit.

The connection between the function $I(x)$ and the given integrand $f(x)$ is extremely important.

THEOREM VIII (ON THE DERIVATIVE OF AN INTEGRAL WITH RESPECT TO ITS UPPER LIMIT). **The derivative of an integral with respect to its upper limit is equal to the integrand:**

$$I'(x) = \left(\int_a^x f(x) dx \right)' = f(x). \quad (\dagger)$$

Proof. We give the argument x the increment Δx . The "incremented" value of the function is now

$$I(x + \Delta x) = \int_a^{x+\Delta x} f(x) dx.$$

Thus

$$\Delta I = I(x + \Delta x) - I(x) = \int_a^{x+\Delta x} f(x) dx - \int_a^x f(x) dx,$$

i.e.

$$\Delta I = \int_x^{x+\Delta x} f(x) dx.$$

We find on applying the mean value theorem (theorem VII) to the last integral:

$$\Delta I = f(\xi)(x + \Delta x - x) = f(\xi) \Delta x,$$

where ξ is a point lying between x and $x + \Delta x$.

We have by definition of the derivative:

$$I'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta I}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(\xi) \Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} f(\xi).$$

But if $\Delta x \rightarrow 0$, then $x + \Delta x$ tends to x ; hence $\xi \rightarrow x$, and since $f(x)$ is a continuous function we have

$$\lim_{\Delta x \rightarrow 0} f(\xi) = \lim_{\xi \rightarrow x} f(\xi) = f(x).$$

This is what we wanted to show.

It also follows from the theorem that

$$d \int_a^x f(x) dx = f(x) dx. \quad (\dagger\dagger)$$

It must be noticed that the results in formulae (†) and (††) do not depend on the notation for the variable of integration; the following equalities hold, for instance:

$$\frac{d}{dx} \int_a^x f(t) dt = f(x), \quad d \int_a^x f(t) dt = f(x) dx.$$

We consider the geometrical interpretation of the theorem. The function $f(x)$ expresses the variable area of the curvilinear trapezoid with variable base $[a, x]$, bounded by the curve $y = f(x)$. It is asserted in theorem VIII that the derivative of the trapezoid area with respect to the abscissa is equal to the ordinate of the curve bounding the

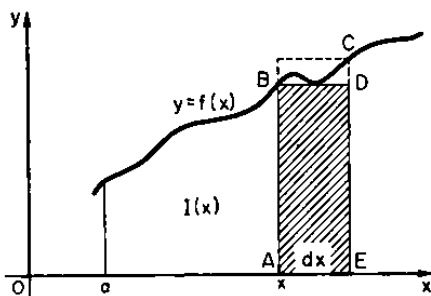


FIG. 117

trapezoid (the segment $AB = f(x)$ in Fig. 117), or that the differential of the trapezoid area is equal to the area of the rectangle with sides equal respectively to the increment of the trapezoid base and the ordinate of the curve at the nearest point.

Let us explain this proposition geometrically. We give the trapezoid base an infinitesimal increment dx ; then the increment $\Delta I(x)$ of the trapezoid area is represented by the area of the infinitesimally narrow curvilinear trapezoid $ABCE$ (Fig. 117). We show that the differential $dI(x)$ of the area is the area of rectangle $ABDE$. To do this, we first show that areas $ABDE$ and $ABCE$ are equivalent infinitesimals. We have:

$$\text{area } ABDE = y \Delta x, \quad \underline{y} \Delta x \leq \text{area } ABCE \leq \bar{y} \Delta x,$$

where $\underline{y}$ and $\bar{y}$ are respectively the least and greatest values of the function $y = f(x)$ in the interval $[x, x + \Delta x]$; hence

$$\frac{y \Delta x}{y \Delta x} \leq \frac{\text{area } ABCE}{\text{area } ABDE} \leq \frac{\bar{y} \Delta x}{y \Delta x}.$$

As $\Delta x \rightarrow 0$, the ratio of the areas tends to unity, because $\underline{y}$ and $\bar{y}$ have the same limit y as $\Delta x \rightarrow 0$ by virtue of the continuity of $f(x)$. Since, in addition, area $ABDE$ ($= y \Delta x$) is proportional to Δx , it is in fact the differential of the area of the curvilinear trapezoid:

$$dI(x) = f(x)dx, \quad \text{or} \quad \frac{dI(x)}{dx} = f(x).$$

Thus the function $I(x)$ is a primitive (Sec. 69) of the given function $f(x)$. Apropos of this, the existence theorem for the integral leads to the theorem:

Every continuous function has a primitive, i.e., in accordance with the theorem of Sec. 69, it has an infinite set of primitives, which differ from each other only by a constant term.

Theorem VIII shows that the application of the operation of integration to any function (with variable upper limit) followed by the operation of differentiation on the result (with respect to this limit) leaves the function unchanged.

(We shall find in Sec. 93 the result of successive application of these operations in the reverse order: first differentiation, then integration.)

These propositions link up the main operations of analysis—differentiation and integration—in such a way that they are in a sense inverse operations: *they so to speak cancel each other when performed consecutively.*

93. The Newton—Leibniz formula.

THEOREM. The value of a definite integral is equal to the difference of the values of any primitive of the integrand at the upper and lower limits:

$$\int_a^b f(x) dx = F(b) - F(a), \quad F'(x) = f(x). \quad (*)$$

Equation (*) is called the *Newton—Leibniz formula*.

In other words,

The value of a definite integral is equal to the increment of any primitive of the integrand over the interval of integration.

Proof. We take the function

$$I(x) = \int_a^x f(x) dx;$$

since it is a primitive of the function $f(x)$, in accordance with the theorem of Sec. 69 it must be sought among the functions $F(x) + C$, where $F(x)$ is any primitive of $f(x)$.

Thus

$$I(x) = F(x) + C_1,$$

where C_1 is some definite constant. To find it, we use the second property of function $I(x)$, viz. that $I(a) = 0$. Thus $F(a) + C_1 = 0$, i.e. $C_1 = -F(a)$. Hence

$$I(x) = \int_a^x f(x) dx = F(x) - F(a);$$

when $x = b$ we obtain the required equation (*).

The difference between the values of the function is often written as

$$F(b) - F(a) = F(x) \Big|_a^b.$$

The vertical line with upper and lower indices, standing to the right of the symbol of the function and known as the double substitution sign, indicates that we have to subtract from the value of the function at the upper index its value at the lower index.

On using this notation, formula (*) can be written as

$$\int_a^b f(x) dx = F(x) \Big|_a^b,$$

where $F'(x) = f(x)$.

The Newton-Leibniz formula gives us a remarkable key to the evaluation of definite integrals. It enables us to find the definite integral with the aid of the primitive of a function, and without the need for summation. We take some simple examples for illustration.

Since $x^{k+1}/(k+1)$ is a primitive of x^k , we have

$$\int_a^b x^k dx = \frac{x^{k+1}}{k+1} \Big|_a^b = \frac{b^{k+1} - a^{k+1}}{k+1} \quad (k \neq -1).$$

We have easily obtained the formula which needed great effort when we deduced it (and only for positive integral k) from the actual definition of the integral, using summation (Sec. 87).

One of the primitives of $\sin x$ is $-\cos x$, so that

$$\int_a^b \sin x dx = -\cos x \Big|_a^b = -\cos b + \cos a.$$

Since one of the primitives of $1/x$ is $\ln x$, we have

$$\int_a^b \frac{1}{x} dx = \ln x \Big|_a^b = \ln b - \ln a = \ln \frac{b}{a}.$$

One of the primitives of e^x is e^x , so that

$$\int_a^b e^x dx = e^x \Big|_a^b = e^b - e^a.$$

If we take any other primitives of the integrands (i.e. differing from the above by a constant term), we evidently obtain the same results.

We now observe that, since $F(x)$ is a primitive of $F'(x)$, we have

$$\int_a^x F'(x) dx = F(x) - F(a),$$

or, written conventionally,

$$\int_a^x dF(x) = F(x) - F(a).$$

This is a slightly different form of the Newton-Leibniz formula, enabling the basic theorem of this article to be stated as:

The increment of a function differentiable in an interval is equal to the definite integral over this interval of the differential of the function.

The formulae

$$\left(\int_a^x f(x) dx \right)' = f(x), \quad \int_a^x f'(x) dx = f(x) - f(a) \quad (**)$$

establish the strict nature of the connection between the concepts of integral and derivative.

This connection fully corresponds to the connection which holds between any other mutually inverse operations. Either of our two operations of analysis can be reckoned as direct; the other, its inverse, is then defined on the basis of relationships(**). Since differentiation of functions is the simpler operation, it is regarded as the direct operation.

Finally, an extremely important result of the present chapter is our conclusion that *definite integration of functions amounts to finding the primitives of these functions.*

REMARK. It is now very easily shown that the path s , found by us in Sec. 85, II with the aid of the integral (with respect to the velocity v), in fact coincides with the path from which we started when defining velocity. Let the path s be given as a function of time by: $s = F(t)$. Then $v = f(t) = F'(t)$ (see Sec. 42). Hence, in accordance with Sec. 85 (and Sec. 86):

$$s = \int_0^T F'(t) dt,$$

whence we find by the Newton-Leibniz formula (taking $F(0) = 0$):

$$s = F(T).$$

Similarly, we have in the problem on mass (Sec. 85, III):

$$m = \int_0^s \varphi(s) ds$$

where $\varphi(s) = \Phi'(s)$ (see Sec. 42), and $m = \Phi(s)$. On substituting in the integral $\Phi'(s)$ instead of $\varphi(s)$ (and taking $\Phi(0) = 0$), we get by the Newton-Leibniz formula:

$$m = \int_0^s \Phi'(s) ds = \Phi(s).$$

THE INDEFINITE INTEGRAL. THE INTEGRAL CALCULUS

1. The Indefinite Integral and Indefinite Integration

94. The indefinite integral. Basic table of integrals.

Definition. The operation of finding primitives is called *indefinite integration*, whilst the expression embracing the set of all primitives of a given function $f(x)$ is called the *indefinite integral of $f(x)$* and is written as

$$\int f(x) dx.$$

The function $f(x)$ is called the integrand, and $f(x) dx$ the integrand expression, x being the variable of integration. In this chapter it will always be assumed that the integrand $f(x)$ is continuous for the x in question.

Since it was shown in Sec. 69 that any primitive differs from a given primitive by a constant, we have

$$\int f(x) dx = F(x) + C,$$

where $F(x)$ is any primitive of $f(x)$ and C is an arbitrary constant.

Indefinite integration is the operation inverse to differentiation. We find with the aid of differentiation the derivative of a given function, whilst we find with the aid of indefinite integration the primitive of any given derivative.

The symbol for the indefinite integral differs from that for the definite integral (see Chapter V) only in the absence of the limits of integration. The choice of such a symbol for denoting the result of the inverse operation to differentiation (and the operation itself) is fully justified by the connection between the primitive and the definite integral as expressed by the Newton-Leibniz formula (Sec. 93).

To find the indefinite integral of a function means to find all its primitives (for which it is sufficient to find one of them). In fact,

we speak of "indefinite integration" because no indication is supplied of the particular primitive we have in mind.

The graph of a primitive of a function $f(x)$ is called an integral curve of the function. Obviously, we get any other integral curve simply by a parallel displacement of the given curve in the direction of the axis of ordinates, by an amount representing the added term C . The geometrical representation of the indefinite integral is thus the set of all integral curves, obtained on continuous parallel displacement of one of them in the direction of Oy from $-\infty$ to $+\infty$ (Fig. 118).

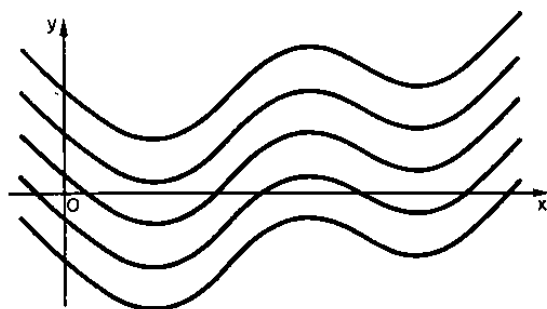


FIG. 118

We have by the definition of indefinite integral:

$$\left(\int f(x) dx\right)' = f(x) \quad \text{or} \quad d \int f(x) dx = f(x) dx,$$

and

$$\int f'(x) dx = f(x) + C \quad \text{or} \quad \int df(x) = f(x) + C.$$

The symbols of the differential and of the indefinite integral cancel each other when applied consecutively (if we ignore the constant term in the formulae of the second line).

In future, whenever there is no danger of confusion, we shall understand by the terms "integral" and "integration" the "indefinite integral" and "indefinite integration".

In the simplest cases we can integrate a function, or as we often say, find its integral, by simple inversion of the suitable differentiation formula.

We shall give here the integration formulae obtained by inversion of the basic formulae of the differential calculus.

Basic table of integrals.

$$\begin{aligned}
\int x^k dx &= \frac{x^{k+1}}{k+1} + C, & k \neq -1. & \quad \int \sin x dx = -\cos x + C. \\
\int \frac{dx}{x} &= \ln|x| + C^*. & & \quad \int \frac{dx}{\cos^2 x} = \tan x + C. \\
\int a^x dx &= \frac{a^x}{\ln a} + C. & & \quad \int \frac{dx}{\sin^2 x} = -\cot x + C. \\
\int e^x dx &= e^x + C. & & \quad \int \frac{dx}{1+x^2} = \arctan x + C. \\
\int \cos x dx &= \sin x + C. & & \quad \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C.
\end{aligned}$$

This table must be learnt by heart.

95. Elementary rules for integration.

THEOREM I. The integral of a sum of a finite number of functions is equal to the sum of the integrals of the separate functions:

$$\begin{aligned}
\int [f(x) + \varphi(x) + \dots + \psi(x)] dx \\
= \int f(x) dx + \int \varphi(x) dx + \dots + \int \psi(x) dx.
\end{aligned}$$

Proof. Let $F(x), \Phi(x), \dots, \Psi(x)$ be primitives of $f(x), \varphi(x), \dots, \psi(x)$ respectively. Then

$$[F(x) + \Phi(x) + \dots + \Psi(x)]' = f(x) + \varphi(x) + \dots + \psi(x)$$

i.e.

$$\int [f(x) + \varphi(x) + \dots + \psi(x)] dx = F(x) + \Phi(x) + \dots + \Psi(x) + C. (*)$$

But

$$\begin{aligned}
\int f(x) dx &= F(x) + A, & \int \varphi(x) dx &= \Phi(x) + B, \dots, \\
& & \int \psi(x) dx &= \Psi(x) + D,
\end{aligned}$$

where $A, B, \dots, D$ are arbitrary constants. On adding all the last equalities, we get

$$\begin{aligned}
\int f(x) dx + \int \varphi(x) dx + \dots + \int \psi(x) dx \\
= F(x) + \Phi(x) + \dots + \Psi(x) + A + B + \dots + D. (**)
\end{aligned}$$

* In this formula x can be either positive or negative; the only impermissible case is when x varies in an interval containing $x = 0$, since the integrand then ceases to be continuous. If $x > 0$, $1/x$ is a primitive of $1/x$; if $x < 0$, putting $z = -x (> 0)$, we find $dx/x = dz/z$, and the primitive of $1/z$ will be $\ln z$, i.e. $\ln|x|$. Thus the formula embraces both cases.

The sum of the arbitrary constants $A + B + \dots + D$ is again an arbitrary constant, which can be denoted by the single letter C . On comparing(*) and (**), we arrive at the desired result†.

THEOREM II. A constant factor in the integrand can be taken outside the integral symbol:

$$\int c f(x) dx = c \int f(x) dx.$$

Proof. If $F'(x) = f(x)$, then $[cF(x)]' = cf(x)$ and

$$\int f(x) dx = F(x) + C, \quad \int c f(x) dx = cF(x) + C_1.$$

The arbitrary constant C_1 can be written as cC , where C is again an arbitrary constant. Hence

$$\int c f(x) dx = c[F(x) + C] = c \int f(x) dx.$$

This is what we had to prove.

Example.

$$\begin{aligned} \int (2 \sin x - 3 \cos x) dx &= 2 \int \sin x dx - 3 \int \cos x dx \\ &= -2 \cos x - 3 \sin x + C. \end{aligned}$$

Although every intermediate integration gives an arbitrary constant, only one appears in the final result because a sum of arbitrary constants is itself an arbitrary constant.

THEOREM III. Every integration formula retains its form on replacing the independent variable by any differentiable function of the variable, i.e. if

$$\int f(x) dx = F(x) + C, \quad \text{then also} \quad \int f(u) du = F(u) + C,$$

where $u = \varphi(x)$ is any differentiable function of x .

Proof. The fact that

$$\int f(x) dx = F(x) + C \quad \text{implies:} \quad f(x) = F'(x).$$

We now take a function $F(u) = F[\varphi(x)]$; we have for its differential, by virtue of the theorem on the invariance of the form of the first differential (Sec. 52):

$$dF(u) = F'(u) du = f(u) du.$$

† The left-hand sides of (*) and (**) differ from the sum $F(x) + \Phi(x) + \dots + \Psi(x)$ by an arbitrary constant, so that they express the same set of functions (viz. the set of primitives of the integrand).

Hence

$$\int f(u) du = \int dF(u) = F(u) + C.$$

This rule is extremely important. The basic table of integrals is valid, by virtue of the rule, independently of whether the variable of integration is the independent variable or some differentiable function of it; the table is thus extended considerably.

Example. Let us find $\int 2x e^{x^2} dx$. On observing that $2x$ is the derivative of x^2 , we can rewrite the integral as

$$\int 2x e^{x^2} dx = \int e^{x^2} d(x^2) = \int e^u du,$$

where we have put $u = x^2$. The last integral is familiar; it is equal to $e^u + C$, i.e.

$$\int 2x e^{x^2} dx = e^{x^2} + C.$$

As this example shows, we must try to transform the integrand in such a way that it becomes an integrand of an integral belonging to the basic table.

Notice that the correctness of the integration can always be checked by differentiating the result.

96. Examples. We consider some typical and important examples.

(1) $\int \sin 5x dx$. We multiply and divide the integral by 5 and take the factor 5 outside the integral symbol:

$$\int \sin 5x dx = \frac{1}{5} \int \sin 5x \cdot 5 dx = \frac{1}{5} \int \sin 5x d(5x).$$

Putting $5x = u$, we arrive at an integral of the basic table:

$$\int \sin 5x dx = \frac{1}{5} \int \sin u du = -\frac{1}{5} \cos u + C = -\frac{1}{5} \cos 5x + C.$$

We can find in the same way e.g.

$$\begin{aligned} \int e^{-3x} dx &= -\frac{1}{3} \int e^{-3x} d(-3x) = -\frac{1}{3} \int e^u du \\ &= -\frac{1}{3} e^u + C = -\frac{1}{3} e^{-3x} + C. \end{aligned}$$

(2) $\int (2x - 1)^{100} dx$. On multiplying and dividing by 2 and observing that $2 dx = d(2x - 1)$, we get:

$$\int (2x - 1)^{100} dx = \frac{1}{2} \int (2x - 1)^{100} d(2x - 1).$$

Putting $2x - 1 = u$, we obtain:

$$\int (2x - 1)^{100} dx = \frac{1}{2} \int u^{100} du = \frac{1}{2} \frac{u^{101}}{101} + C = \frac{1}{202} (2x - 1)^{101} + C.$$

The reader will appreciate all the advantages of this integration if he compares it with integration of the polynomial obtained by expanding the binomial to the 100th (!) degree by Newton's formula.

(3) $\int x^2 \sqrt{4 - 3x^3} dx$. We transform the integral:

$$\begin{aligned} \int x^2 \sqrt{4 - 3x^3} dx &= -\frac{1}{9} \int \sqrt{4 - 3x^3} \cdot (-9x^2) dx \\ &= -\frac{1}{9} \int \sqrt{4 - 3x^3} d(4 - 3x^3). \end{aligned}$$

The last transformation is based on the fact that $d(4 - 3x^3) = -9x^2 dx$. Putting $4 - 3x^3 = u$, we get:

$$\begin{aligned} \int x^2 \sqrt{4 - 3x^3} dx &= -\frac{1}{9} \int u^{\frac{1}{2}} du = -\frac{2}{27} u^{\frac{3}{2}} + C \\ &= -\frac{2}{27} (4 - 3x^3) \sqrt{4 - 3x^3} + C. \end{aligned}$$

As skill in integration develops it becomes unnecessary to write down all the intermediate working and notations. The major part is done in one's head.

(4) $\int 2 dx / (3x - 1)$. We transform the integral thus:

$$\int \frac{2 dx}{3x - 1} = \frac{2}{3} \int \frac{3 dx}{3x - 1} = \frac{2}{3} \int \frac{(3x - 1)'}{3x - 1} dx = \frac{2}{3} \int \frac{d(3x - 1)}{3x - 1}.$$

On writing u for $3x - 1$, we arrive at the tabulated integral

$$\int \frac{2 dx}{3x - 1} = \frac{2}{3} \int \frac{du}{u} = \frac{2}{3} \ln |u| + C = \frac{2}{3} \ln |3x - 1| + C.$$

In general, if the numerator of the integrand is the derivative of the denominator, the integral is equal to the logarithm of the absolute value of the denominator. In fact,

$$\int \frac{f'(x) dx}{f(x)} = \int \frac{df(x)}{f(x)} = \int \frac{du}{u} = \ln |u| + C = \ln |f(x)| + C.$$

For instance:

$$\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \ln |\sin x| + C.$$

(5) $\int (3x + 2)/(2x - 1) dx$. On dividing the numerator by the denominator, we get 1.5 in the quotient and 3.5 in the remainder. Thus

$$\begin{aligned}\int \frac{3x + 2}{2x - 1} dx &= \int \left(1.5 + \frac{3.5}{2x - 1} \right) dx = \int 1.5 dx + \int \frac{3.5}{2x - 1} dx \\ &= 1.5x + 1.75 \ln |2x - 1| + C.\end{aligned}$$

(6) $\int dx/(x^2 - 1)$. We write the denominator as the product of two linear factors:

$$\int \frac{dx}{x^2 - 1} = \int \frac{dx}{(x - 1)(x + 1)}.$$

We write the numerator as $1 = \frac{1}{2}[(x + 1) - (x - 1)]$, so that

$$\begin{aligned}\int \frac{dx}{x^2 - 1} &= \frac{1}{2} \int \frac{(x + 1) - (x - 1)}{(x - 1)(x + 1)} dx \\ &= \frac{1}{2} \int \frac{x + 1}{(x - 1)(x + 1)} dx - \frac{1}{2} \int \frac{x - 1}{(x - 1)(x + 1)} dx,\end{aligned}$$

and furthermore

$$\begin{aligned}\int \frac{dx}{x^2 - 1} &= \frac{1}{2} \int \frac{dx}{x - 1} - \frac{1}{2} \int \frac{dx}{x + 1} \\ &= \frac{1}{2} \ln |x - 1| - \frac{1}{2} \ln |x + 1| + C = \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| + C.\end{aligned}$$

We can always proceed in this way when the numerator of the integrand is constant and the denominator is a quadratic expression which can be expanded into linear factors. For example:

$$\begin{aligned}\int \frac{1}{x^2 - 5x + 6} dx &= \int \frac{dx}{(x - 2)(x - 3)} \\ &= \int \frac{(x - 2) - (x - 3)}{(x - 2)(x - 3)} dx = \ln \left| \frac{x - 3}{x - 2} \right| + C\end{aligned}$$

(7) $\int \sin x \cos x dx$. We have by the familiar trigonometric formula:

$$\int \sin x \cos x dx = \frac{1}{2} \int \sin 2x dx = \frac{1}{4} \int \sin 2x d(2x) = -\frac{1}{4} \cos 2x + C.$$

This integral can be found in another way:

$$\int \sin x \cos x \, dx = \int \sin x \, d(\sin x) = \frac{\sin^2 x}{2} + C,$$

or

$$\int \sin x \cos x = - \int \cos x \, d(\cos x) = - \frac{\cos^2 x}{2} + C.$$

It can be seen that three essentially different answers have been obtained for the same integral:

$$-\frac{1}{4} \cos 2x + C, \quad \frac{1}{2} \sin^2 x + C, \quad -\frac{1}{2} \cos^2 x + C.$$

But C , being an arbitrary constant, can be written say as $C = 1/4 + C'$, where C' is also an arbitrary constant. The first expression now turns into the second:

$$-\frac{1}{4} \cos 2x + \frac{1}{4} + C' = \frac{1}{4}(1 - \cos 2x) + C' = \frac{1}{2} \sin^2 x + C'.$$

It can be verified similarly that any two of our expressions only differ by a constant, i.e. any one of them gives all the primitives of $\sin x \cos x$.

(8) $\int \cos^3 x \, dx$. We obtain successively:

$$\begin{aligned} \int \cos^3 x \, dx &= \int \cos^2 x \, d(\sin x) = \int (1 - \sin^2 x) \, d(\sin x) \\ &= \int d(\sin x) - \int \sin^2 x \, d(\sin x) = \sin x - \frac{1}{3} \sin^3 x + C. \end{aligned}$$

(9) $\int dx/\sin x$. We write $\sin x$ as a function of half the argument:

$$\int \frac{dx}{\sin x} = \int \frac{dx}{2 \sin \frac{x}{2} \cos \frac{x}{2}}.$$

Further, we divide the numerator and denominator of the integrand by $\cos^2 \frac{1}{2}x$ and take $\frac{1}{2}$ inside the differential sign. Thus

$$\int \frac{dx}{\sin x} = \int \frac{\frac{1}{\cos^2 \frac{x}{2}} d\left(\frac{x}{2}\right)}{\tan \frac{x}{2}} = \int \frac{d\left(\tan \frac{x}{2}\right)}{\tan \frac{x}{2}} = \ln \left| \tan \frac{x}{2} \right| + C.$$

Another method can be used. On replacing 1 in the numerator by $\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}$, we get

$$\begin{aligned} \int \frac{dx}{\sin x} &= \int \frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}}{\sin \frac{x}{2} \cos \frac{x}{2}} d\left(\frac{x}{2}\right) = \int \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} d\left(\frac{x}{2}\right) + \\ &+ \int \frac{\cos \frac{x}{2}}{\sin \frac{x}{2}} d\left(\frac{x}{2}\right) = -\ln \left| \cos \frac{x}{2} \right| + \ln \left| \sin \frac{x}{2} \right| + C = \ln \left| \tan \frac{x}{2} \right| + C. \end{aligned}$$

(10) $\int dx/(a^2 + x^2)$. We have:

$$\begin{aligned} \int \frac{dx}{a^2 + x^2} &= \frac{1}{a^2} \int \frac{dx}{1 + \left(\frac{x}{a}\right)^2} = \int \frac{d\left(\frac{x}{a}\right)}{1 + \left(\frac{x}{a}\right)^2} = \frac{1}{a} \int \frac{du}{1 + u^2} \\ &= \frac{1}{a} \arctan u + C = \frac{1}{a} \arctan \frac{x}{a} + C. \end{aligned}$$

We can find $\int dx/\sqrt{a^2 - x^2}$ in the same way:

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{a} \int \frac{dx}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} = \int \frac{d\left(\frac{x}{a}\right)}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} = \arcsin \frac{x}{a} + C.$$

There is no need to memorize these results. The important thing is to note the methods of transformation.

2. Basic Methods of Integration

97. Integration by parts. Since integration is the inverse operation to differentiation, for every differentiation rule there must be a corresponding integration rule. The three rules for integration proved in art. 1 correspond to the rules for differentiating a sum, the product of a function and a constant, and a function of a function.

We now turn to the so-called method of integration by parts, which is obtained by inversion of the formula for differentiating the product of two functions.

Let u and v be differentiable functions of x . We have:

$$d(uv) = u dv + v du,$$

whence

$$u dv = d(uv) - v du.$$

We obtain by integrating both sides of the last equation:

$$\int u dv = \int d(uv) - \int v du,$$

or

$$\int u dv = uv - \int v du. \quad (*)$$

This is in fact the formula for integration by parts. We do not write down the arbitrary constant in the integration of $d(uv)$, but associate it with the arbitrary constant of the second integration not yet performed on the right-hand side of (*).

Integration by parts consists in somehow writing the integrand expression $f(x) dx$ as the product of two factors u and dv (the latter must include dx), and replacing the integration, in accordance with formula (*), by two integrations: (1) for finding v from the expression for dv ; (2) for finding the integral of $v du$. It may happen that these two integrations are easily carried out, whereas the given integral is difficult to find directly.

Examples. (1) $\int x e^x dx$. We take $e^x dx$ as dv , x as u , i.e.

$$\left. \begin{array}{l} e^x dx = dv, \\ x = u, \end{array} \right\} \text{whence} \quad \left\{ \begin{array}{l} v = \int e^x dx = e^x, \\ du = dx. \end{array} \right.$$

We get by formula (*):

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + C = e^x(x - 1) + C.$$

The given unfamiliar integral is seen to be reduced here to a familiar one.

(2) $\int \ln x dx$. We put:

$$\left. \begin{array}{l} dx = dv, \\ \ln x = u, \end{array} \right\} \text{whence} \quad \left\{ \begin{array}{l} v = x, \\ du = \frac{dx}{x}. \end{array} \right.$$

Thus

$$\int \ln x dx = x \ln x - \int dx = x \ln x - x + C = x(\ln x - 1) + C.$$

As skill develops it becomes unnecessary to write down all the intermediate steps.

It is sometimes necessary to carry out integration by parts several times in succession in order to obtain the required result.

(3) $\int x^2 \cos x \, dx$. We put:

$$\begin{array}{l|l} \cos x \, dx = dv, & v = \sin x, \\ x^2 = u, & du = 2x \, dx. \end{array}$$

Formula (*) gives:

$$\int x^2 \cos x \, dx = x^2 \sin x - 2 \int x \sin x \, dx.$$

The last integral is again taken by parts. We finally get:

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

By repeated integration by parts we can find the integrals

$$\int x^m \sin x \, dx, \quad \int x^m \cos x \, dx, \quad \int x^m e^x \, dx$$

(m is a positive integer), and thus the integrals

$$\int P(x) \sin x \, dx, \quad \int P(x) \cos x \, dx, \quad \int P(x) e^x \, dx,$$

where $P(x)$ is any polynomial.

Repeated integration by parts sometimes leads to the original integral, in which case either nothing but an identity is obtained (so that the integration has been done pointlessly) or else we get an equation from which an expression can be found for the integral. We shall illustrate this by some examples.

(4) $\int e^x \cos x \, dx$. We put:

$$\begin{array}{l|l} e^x \, dx = dv, & v = e^x, \\ \cos x = u, & du = -\sin x \, dx. \end{array}$$

Hence

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx.$$

On again integrating by parts we get†:

$$\begin{array}{l|l} e^x \, dx = dv, & v = e^x, \\ \sin x = u, & du = \cos x \, dx. \end{array}$$

We now have: $\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$, and we have arrived at the original integral. On substituting the expression found

† If we take $\sin x \, dx$ as dv here, we arrive at the identity

$$\int e^x \cos x \, dx = \int e^x \cos x \, dx.$$

in the result of the first operation, we get:

$$\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx;$$

on taking over the integral from the right-hand side to the left, we find:

$$2 \int e^x \cos x \, dx = e^x (\cos x + \sin x) + C$$

i.e.

$$\int e^x \cos x \, dx = \frac{1}{2} e^x (\cos x + \sin x) + C.$$

(5) $\int \sqrt{1-x^2} \, dx$. We multiply and divide the integrand by $\sqrt{1-x^2}$ and split the integral into two:

$$\int \frac{1-x^2}{\sqrt{1-x^2}} \, dx = \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x^2 dx}{\sqrt{1-x^2}}.$$

The first integral on the right is tabulated ($= \arcsin x$); the second is found by parts as follows:

$$\left. \begin{aligned} \frac{x}{\sqrt{1-x^2}} \, dx &= dv, \\ x &= u, \end{aligned} \right| \begin{aligned} v &= -\sqrt{1-x^2}, \\ du &= dx. \end{aligned}$$

Thus

$$\int \frac{\sqrt{1-x^2}}{x^2} \, dx = -x \sqrt{1-x^2} + \int \sqrt{1-x^2} \, dx$$

We thus obtain the equation:

$$\int \sqrt{1-x^2} \, dx = \arcsin x + x \sqrt{1-x^2} - \int \sqrt{1-x^2} \, dx,$$

from which the required integral is found:

$$\int \sqrt{1-x^2} \, dx = \frac{1}{2} (\arcsin x + x \sqrt{1-x^2}) + C.$$

The rule for integration by parts often enables us to carry through an integration. But it is impossible to give an advance recipe for the actual mode of application. Experience shows in which cases there is some point in trying integration by parts and which factor of the integrand is best taken as dv .

98. Change of variable. The method of changing the variable of integration, or of substitution, consists in the following. Suppose we have succeeded in transforming the integrand expression $f(x) \, dx$

to the form

$$f(x) dx = f_1[\varphi(x)]\varphi'(x) dx;$$

then

$$\int f(x) dx = \int f_1[\varphi(x)]\varphi'(x) dx$$

or, on putting $\varphi(x) = u$:

$$\int f(x) dx = \int f_1(u) du.$$

If the primitive of $f_1(u)$ is known and equal to $F(u)$, we have

$$\int f(x) dx = F(u) + C = F[\varphi(x)] + C.$$

Strictly speaking, we have already used this method, in Sec. 95, III.

Thus the method consists in replacing the variable of integration x by another variable u , connected with x by the formula $u = \varphi(x)$.

It is possible to carry out the change by expressing x in terms of u , instead of u in terms of x , i.e. putting

$$x = \psi(u), \quad dx = \psi'(u) du$$

and

$$\int f(x) dx = \int f[\psi(u)]\psi'(u) du. \quad (\dagger)$$

If the last integral can somehow be found (let it be equal to $F(u) + C$), the expression for the given integral is obtained by returning to the variable x , i.e. by substituting the expression for x in place of u in $F(u) + C$.

Equation ($\dagger$) can also easily be verified by taking the derivative with respect to x of both sides and recalling that $\psi(u) = x$.

We used the rule for change of variable in Sec. 95 by *first transforming the integrand expression* (leading to a familiar type), *then introducing a new variable in accordance with* $u = \varphi(x)$. The reverse order may be recommended as more convenient in complicated cases: *first choose the substitution formula* $u = \varphi(x)$ *or* $x = \psi(u)$, *then suitably transform the integrand expression*. No general rules can be given for the choice of substitution.

Examples. (1) $\int x^2 \sqrt{4 - 3x^3} dx$. We put $\sqrt{4 - 3x^3} = u$, i.e. $4 - 3x^3 = u^2$.

Differentiation gives: $-9x^2 dx = 2u du$ and $x^2 dx = -2u du/9$. Hence

$$\begin{aligned} \int x^2 \sqrt{4 - 3x^3} dx &= -\frac{2}{9} \int u^2 du = -\frac{2}{27} u^3 + C \\ &= -\frac{2}{27} (4 - 3x^3) \sqrt{4 - 3x^3} + C. \end{aligned}$$

This example has already been solved (Sec. 96, example 3) by direct transformation of the integrand.

(2) $\int \sin x \cos x dx$. We put $\sin x = u$. Then $\cos x dx = du$, i.e.

$$\int \sin x \cos x dx = \int u du = \frac{u^2}{2} + C = \frac{\sin^2 x}{2} + C.$$

This integral was also found in Sec. 96. It is an example of the general type of integral:

$$\int \sin n x \cos m x dx,$$

where n and m are integers. We have:

$$\sin n x \cos m x = \frac{1}{2} [\sin(n-m)x + \sin(n+m)x]$$

i.e. (with $n \neq m$),

$$\begin{aligned} \int \sin n x \cos m x dx &= \frac{1}{2} \int \sin(n-m)x dx + \frac{1}{2} \int \sin(n+m)x dx \\ &= -\frac{1}{2(n-m)} \cos(n-m)x - \frac{1}{2(n+m)} \cos(n+m)x + C. \end{aligned}$$

(3) $\int \sin^3 x \cos^2 x dx$. We substitute $\cos x = u$, $-\sin x dx = du$. This gives:

$$\begin{aligned} \int \sin x (1 - \cos^2 x) \cos^2 x dx &= -\int (1 - u^2) u^2 du = -\frac{u^3}{3} + \frac{u^5}{5} + C \\ &= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C. \end{aligned}$$

Other methods often need to be used in addition to change of variable (e.g. integration by parts) in order to bring the problem to a conclusion.

(4) $\int x^5 e^{x^3} dx$. We put $x^3 = u$; then $3x^2 dx = du$, i.e.

$$\int x^2 \cdot x^3 e^{x^3} dx = \frac{1}{3} \int u e^u du,$$

and this integral is found by parts, as we know from Sec. 97, example 1. We obtain:

$$\int x^5 e^{x^3} dx = \frac{1}{3} e^u (u - 1) + C = \frac{1}{3} e^{x^3} (x^3 - 1) + C.$$

Substitutions of the type $\varphi(x) = u$ have been used in the above examples. We now take examples in which substitutions of the type $x = \psi(u)$ are convenient. We start with the useful "trigonometric substitutions".

(5) $\int \sqrt{1-x^2} dx$. We put $x = \sin u$, $dx = \cos u du$.

We find on substituting:

$$\int \sqrt{1-x^2} dx = \int \cos^2 u du.$$

The last integral is found either by parts or by replacing $\cos^2 u$ with $\frac{1}{2}(1 + \cos 2u)$. We have:

$$\begin{aligned} \int \sqrt{1-x^2} dx &= \frac{1}{2} \left(u + \frac{1}{2} \sin 2u \right) + C = \frac{1}{2} (u + \sin u \cos u) \\ &= \frac{1}{2} (\arcsin x + x \sqrt{1-x^2}) + C. \end{aligned}$$

This integral was found by a different method in Sec. 97 (example 5).

(6) $\int dx/\sqrt{x^2+1}$. We put

$$x = \tan u, dx = du/\cos^2 u.$$

We find on substituting:

$$\int \frac{dx}{\sqrt{x^2+1}} = \int \frac{du}{\cos u}.$$

The last integral can be found by various methods, (cf. example 9, Sec. 96):

$$\begin{aligned} \int \frac{du}{\cos u} &= 2 \int \frac{1}{\cos^2 \frac{u}{2} - \sin^2 \frac{u}{2}} d\left(\frac{u}{2}\right) = 2 \int \frac{\frac{1}{\cos^2 \frac{u}{2}} d\left(\frac{u}{2}\right)}{1 - \tan^2 \frac{u}{2}} \\ &= 2 \int \frac{dv}{1-v^2}, \end{aligned}$$

where $v = \tan \frac{1}{2} u$. We have further (cf. example 6, Sec. 96):

$$2 \int \frac{dv}{1-v^2} = \ln \left| \frac{1+v}{1-v} \right| + C;$$

since

$$v = \tan \frac{u}{2} = \frac{\tan u}{1 + \sqrt{\tan^2 u + 1}} = \frac{x}{1 + \sqrt{x^2 + 1}},$$

then

$$\begin{aligned} \frac{1+v}{1-v} &= \frac{1 + \frac{x}{1 + \sqrt{x^2 + 1}}}{1 - \frac{x}{1 + \sqrt{x^2 + 1}}} = \frac{(1 + \sqrt{x^2 + 1} + x)(1 + \sqrt{x^2 + 1} + x)}{(1 + \sqrt{x^2 + 1} - x)(1 + \sqrt{x^2 + 1} + x)} \\ &= \frac{\sqrt{x^2 + 1} + x^2 + 1 + x + x\sqrt{x^2 + 1}}{1 + \sqrt{x^2 + 1}} \\ &= \frac{\sqrt{x^2 + 1}(1 + \sqrt{x^2 + 1}) + x(1 + \sqrt{x^2 + 1})}{1 + \sqrt{x^2 + 1}} = x + \sqrt{x^2 + 1}. \end{aligned}$$

Thus

$$\int \frac{dx}{\sqrt{x^2 + 1}} = \ln(x + \sqrt{x^2 + 1}) + C.$$

We do not use the absolute value sign here, since the expression $x + \sqrt{x^2 + 1}$ is always positive with the positive sign of the radical.

(7) Similarly, we find by substituting $x = 1/\sin u$, that

$$\int \frac{dx}{\sqrt{x^2 - 1}} = \ln|x + \sqrt{x^2 - 1}| + C.$$

The results of the last two examples can be combined as

$$\int \frac{dx}{\sqrt{x^2 \pm 1}} = \ln|x + \sqrt{x^2 \pm 1}| + C.$$

The integrals of the last two examples are found very simply if use is made of "hyperbolic substitutions". The reader is recommended to carry out the relevant working.

The trigonometric substitutions also enable the following integrals to be found easily:

$$(8) \quad \int \sqrt{x^2 - 1} dx = \frac{1}{2} [x\sqrt{x^2 - 1} - \ln|x + \sqrt{x^2 - 1}|] + C.$$

$$(9) \quad \int \frac{dx}{(l^2 + x^2)\sqrt{l^2 + x^2}}.$$

Here it is convenient to put

$$x = l \tan u, \quad dx = l \, du / \cos^2 u,$$

which gives:

$$\begin{aligned} \int \frac{l \, du}{\cos^2 u \cdot \frac{l^2}{\cos^2 u} \cdot \frac{l}{\cos u}} &= \frac{1}{l^2} \int \cos u \, du = \frac{1}{l^2} \sin u + C \\ &= \frac{1}{l^2} \sin \arctan \frac{x}{l}. \end{aligned}$$

But $\sin \alpha = \tan \alpha / \sqrt{1 + \tan^2 \alpha}$, so that

$$\sin \arctan x/l = x/\sqrt{l^2 + x^2}.$$

Consequently

$$\int \frac{dx}{(l^2 + x^2) \sqrt{l^2 + x^2}} = \frac{1}{l^2} \frac{x}{\sqrt{l^2 + x^2}} + C.$$

The process of integration usually consists in the suitable application of the methods we have discussed (algebraic transformations of the integrand expression, integration by parts, integration by substitution) whereby we reduce the given integral to a familiar type.

The reader must be warned against unconsidered trial of various methods. The solution of a sufficient number of examples should serve to develop a skill in rapidly finding the shortest way to obtain an integral.

We can now supplement the table of basic formulae of the integral calculus (Sec. 94) with four more formulae in common use:

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + C,$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C,$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln |x + \sqrt{x^2 \pm a^2}| + C,$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C.$$

3. Basic Classes of Integrable Functions*

99. Linear rational functions. The most important general class of integrable functions is the class of all rational functions. A fairly simple sequence of operations (algorithm) can in fact be given with the aid of which we can find the elementary function which is the integral of a given rational function. This algorithm is based on familiar algebraic facts, and we shall start our discussion with some details of an algebraic nature.

FUNDAMENTAL THEOREM OF ALGEBRA. Every algebraic equation (n is a positive integer):

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0 \quad (*)$$

has precisely the same number of roots (real or complex) as its degree n ; if the coefficients $a_1, \dots, a_n$ are real numbers, for every complex root there is a corresponding conjugate root†.

This leads to the following proposition.

THEOREM. Every polynomial of the n -th degree can be written as the product of n linear factors:

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = (x - a_1)(x - a_2) \cdots (x - a_n),$$

where $a_1, a_2, \dots, a_n$ are the roots of equation (*).

If k of the n roots are equal: $\alpha_1 = \alpha_2 = \cdots = \alpha_k = \alpha$ (i.e. α is a root of multiplicity k), we can write the single factor $(x - \alpha)^k$ in the product instead of the corresponding k factors; if the coefficients of the equation are real, each pair of factors corresponding to conjugate complex roots can be replaced by a single quadratic expression $x^2 + px + q$, where p and q are real numbers**.

Therefore

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = (x - \alpha)^k \cdots (x^2 + px + q)^l, \quad (**)$$

* The term "integrable function" is used here for brevity, in the sense that the indefinite integral of it can be expressed by an elementary function. In analysis, and especially in what is called the theory of functions, an "integrable function" is taken to mean one for which the definite integral exists (Sec. 86).

† This means that, if $\alpha = c + id$ ($i = \sqrt{-1}$) is a root, then $\bar{\alpha} = c - id$ (c and d are real numbers) is also a root.

** For, $(x - \alpha)(x - \bar{\alpha}) = x^2 - (\alpha + \bar{\alpha})x + \alpha \cdot \bar{\alpha}$, whilst the sum and product of conjugate complex numbers are both real. The replacement of two complex by a single real factor saves us from dealing with complex numbers.

where $k, \dots$ are the multiplicities of the real roots, and $t, \dots$ the multiplicities of the conjugate complex roots.

In order actually to obtain this product (which is called expanding the polynomial in linear and quadratic factors), we have to find all the roots, i.e. solve the corresponding equation (*). This represents a special problem, with which we have been partially concerned in Sec. 76; we shall not dwell further on it and shall assume in future that the expansion of the polynomial in the form (**) has been accomplished somehow or other.

We take the linear rational function

$$\frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials,

$$P(x) = b_0 x^m + b_1 x^{m-1} + \dots + b_{m-1} x + b_m,$$

$$Q(x) = x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n,$$

and suppose that the degree of polynomial $P(x)$ is less than that of $Q(x)$, $m < n$. Let $Q(x)$ be written as a product of real linear and quadratic factors:

$$Q(x) = (x - \alpha)^k \dots (x^2 + p x + q)^t.$$

The following algebraic proposition is important in the integral calculus.

THEOREM. Numbers A, B, C exist such that we have identically:

$$\begin{aligned} \frac{P(x)}{Q(x)} &= \frac{A_k}{(x - \alpha)^k} + \dots + \frac{A_2}{(x - \alpha)^2} + \frac{A_1}{x - \alpha} + \\ &\dots \dots \dots \\ &+ \frac{B_t x + C_t}{(x^2 + p x + q)^t} + \dots + \frac{B_2 x + C_2}{(x^2 + p x + q)^2} + \frac{B_1 x + C_1}{x^2 + p x + q} \quad (\S) \\ &\dots \dots \dots \end{aligned}$$

The fraction $A_k/(x - \alpha)^k$ is called a partial fraction of the first kind, and $(B_t x + C_t)/(x^2 + p x + q)^t$ a partial fraction of the second kind.

The theorem thus asserts the possibility of writing any rational fraction as a sum of partial fractions.

In the expansion of the fraction $P(x)/Q(x)$, for each real k -ple root (α) of $Q(x)$ there are k corresponding partial fractions of the 1st

kind:

$$\frac{A_k}{(x-\alpha)^k}, \frac{A_{k-1}}{(x-\alpha)^{k-1}}, \dots, \frac{A_2}{(x-\alpha)^2}, \frac{A_1}{x-\alpha},$$

whilst for each pair of conjugate complex roots $(\alpha, \bar{\alpha})$ of multiplicity t of $Q(x)$ there are t partial fractions of the 2nd kind:

$$\frac{B_t x + C_t}{(x^2 + px + q)^t}, \frac{B_{t-1} x + C_{t-1}}{(x^2 + px + q)^{t-1}}, \dots, \\ \frac{B_2 x + C_2}{(x^2 + px + q)^2}, \frac{B_1 x + C_1}{x^2 + px + q},$$

where $x^2 + px + q = (x - \alpha)(x - \bar{\alpha})$ and the discriminant $p^2 - 4q < 0$, since the roots α and $\bar{\alpha}$ are complex.

Given the expansion of its denominator into real linear and quadratic factors, the expansion of a rational fraction into partial fractions requires the discovery of all the coefficients A, B, C in expression (§). This can be done, for instance, by the method of undetermined coefficients (see Sec. 77). Having written down expansion (§), the existence of which is known in advance, with undetermined coefficients (letters), we then clear fractions and obtain an identity between the numerator of the fraction, the polynomial $P(x)$, and a polynomial with n unknown coefficients. Being an identity, we can compare coefficients of like powers of x on the left- and right-hand sides, and hence arrive at a system of n equations with n unknowns, from which the latter can be found.

Instead of equating coefficients, another procedure is often adopted: n different particular values of the independent variable are substituted in the identity. It is especially convenient to take as these values zeros of the denominator of the given fraction. We again arrive at n equations with n unknowns. The reader will become familiar with the methods here described for evaluating coefficients A, B, C from the examples of Sec. 100.

If the power m of the numerator of the fraction $P(x)/Q(x)$ is greater than or equal to the power n of the denominator, we divide polynomial $P(x)$ by $Q(x)$ and obtain a polynomial $N(x)$ for the quotient and a polynomial $M(x)$ in the remainder of degree not exceeding $(n - 1)$. Consequently,

$$\frac{P(x)}{Q(x)} = N(x) + \frac{M(x)}{Q(x)}.$$

Integration of $N(x)$ presents no difficulties, so that the problem simply reduces to integration of a fraction in which the degree of the numerator is less than the degree of the denominator.

Thus, let $m < n$. On the basis of the above algebraic theorem, we can now say that our integration problem reduces to the integration of partial fractions.

We take separately the four possible cases:

$$(1) k = 1, (2) k > 1, (3) t = 1, (4) t > 1.$$

The first three are extremely simple, whereas the last requires more complicated working.

$$1. \int \frac{A}{x - \alpha} dx = A \ln |x - \alpha| + C.$$

$$2. \int \frac{A}{(x - \alpha)^k} dx = A \int (x - \alpha)^{-k} dx \\ = \frac{A}{-k + 1} \frac{1}{(x - \alpha)^{k-1}} + C \quad (k > 1).$$

$$3. \int \frac{Bx + C}{x^2 + px + q} dx.$$

We try to arrange for the derivative of the denominator to appear as a term in the numerator: we multiply and divide the integrand by 2, then add and subtract Bp in the numerator. This gives:

$$\int \frac{Bx + C}{x^2 + px + q} dx = \frac{1}{2} \int \frac{2Bx + Bp + 2C - Bp}{x^2 + px + q} dx \\ = \frac{B}{2} \int \frac{2x + p}{x^2 + px + q} dx + \frac{2C - Bp}{2} \int \frac{dx}{x^2 + px + q}.$$

The first integral on the right-hand side is familiar, since the numerator is the derivative of the denominator; as regards the second integral, on completing the square in the denominator:

$$x^2 + px + q = x^2 + 2 \frac{p}{2} x + \frac{p^2}{4} + q - \frac{p^2}{4} = \left(x + \frac{p}{2}\right)^2 + \frac{4q - p^2}{4}$$

and observing that $4q - p^2 > 0$ by hypothesis, we get:

$$\begin{aligned}\int \frac{dx}{x^2 + px + q} &= \int \frac{d\left(x + \frac{p}{2}\right)}{\left(x + \frac{p}{2}\right)^2 + \frac{4q - p^2}{4}} \\ &= \frac{2}{\sqrt{4q - p^2}} \arctan \frac{2\left(x + \frac{p}{2}\right)}{\sqrt{4q - p^2}} + C\end{aligned}$$

(Sec. 96, example 10). Thus

$$\begin{aligned}\int \frac{Bx + C}{x^2 + px + q} dx &= \frac{B}{2} \ln(x^2 + px + q) + \\ &\quad + \frac{2C - Bp}{\sqrt{4q - p^2}} \arctan \frac{2x + p}{\sqrt{4q - p^2}} + C.\end{aligned}$$

4. $\int (Bx + C)/(x^2 + px + q)^t dx$. We obtain as above:

$$\begin{aligned}\int \frac{Bx + C}{(x^2 + px + q)^t} dx &= \frac{B}{2} \int \frac{2x + p}{(x^2 + px + q)^t} dx + \\ &\quad + \frac{2C - Bp}{2} \int \frac{dx}{(x^2 + px + q)^t}.\end{aligned}$$

The first integral on the right-hand side is easily found:

$$\begin{aligned}\int \frac{2x + p}{(x^2 + px + q)^t} dx &= \int \frac{d(x^2 + px + q)}{(x^2 + px + q)^t} \\ &= \frac{1}{-t + 1} \frac{1}{(x^2 + px + q)^{t-1}} + C \quad (t > 1).\end{aligned}$$

In the second integral, we first of all put for simplicity $x + \frac{1}{2}p = \sqrt{\delta}z$, where $\delta = \frac{4q - p^2}{4}$ ($\delta > 0$ by hypothesis). Now:

$$\int \frac{dx}{(x^2 + px + q)^t} = \int \frac{dx}{\left[\left(x + \frac{p}{2}\right)^2 + \frac{4q - p^2}{4}\right]^t} = \frac{\sqrt{\delta}}{\delta^t} \int \frac{dz}{(z^2 + 1)^t}.$$

Integration of the partial fraction in the fourth case thus reduces to finding the integral

$$I_t = \int \frac{dz}{(z^2 + 1)^t}.$$

We use the following method. We add and subtract z^2 in the numerator, and split the integral into two:

$$I_t = \int \frac{(z^2 + 1) - z^2}{(z^2 + 1)^t} dz = \int \frac{dz}{(z^2 + 1)^{t-1}} - \int \frac{z^2 dz}{(z^2 + 1)^t};$$

we find the last integral by parts by putting

$$\frac{z}{(z^2 + 1)^t} dz = dv, \quad z = u;$$

hence

$$\begin{aligned} v &= \frac{1}{2} \int \frac{2z}{(z^2 + 1)^t} dz = \frac{1}{2} \int \frac{d(z^2 + 1)}{(z^2 + 1)^t} \\ &= \frac{1}{2(-t + 1)} \frac{1}{(z^2 + 1)^{t-1}}, \quad du = dz \end{aligned}$$

and

$$\int \frac{z^2 dz}{(z^2 + 1)^t} = \frac{1}{2(-t + 1)} \frac{z}{(z^2 + 1)^{t-1}} - \frac{1}{2(-t + 1)} \int \frac{dz}{(z^2 + 1)^{t-1}}.$$

Thus

$$I_t = \int \frac{dz}{(z^2 + 1)^{t-1}} + \frac{1}{2(t-1)} \frac{z}{(z^2 + 1)^{t-1}} - \frac{1}{2(t-1)} \int \frac{dz}{(z^2 + 1)^{t-1}},$$

i.e.

$$I_t = \frac{1}{2(t-1)} \frac{z}{(z^2 + 1)^{t-1}} + \frac{2t-3}{2(t-1)} \int \frac{dz}{(z^2 + 1)^{t-1}}.$$

The integral at which we have arrived is of the same type as the original, but the power of the denominator is reduced by 1. We can write:

$$I_t = \frac{z}{2(t-1)(z^2 + 1)^{t-1}} + \frac{2t-3}{2(t-1)} I_{t-1}.$$

We have obtained a reduction (or recurrence) formula for finding I_t^* .

In fact, by using this formula, we can at once express I_{t-1} in terms of I_{t-2} , simply by replacing t everywhere in the formula by

* In general, a recurrence formula connects the values of an expression (function) corresponding to different values of some parameter; the parameter on which the expression depends is usually an integer. Thanks to this connection, it is possible to find successively all the values of the expression in terms of certain known values (corresponding more often than not to the initial values of the parameter). Recurrence formulae are quite often encountered in mathematics, in particular in the integral calculus, where they generally owe their origin to integration by parts.

$t - 1$. Similarly, we obtain successively expressions for I_{t-2} in terms of I_{t-3} , I_{t-3} in terms of I_{t-4} , and so on, until we arrive at the familiar integral I_1 :

$$I_1 = \int \frac{dz}{z^2 + 1} = \arctan z + C.$$

Let us sum up. We have seen that the integrals of partial fractions are expressed in terms of elementary functions: rational, logarithmic and inverse trigonometric (arctangent). Any rational fraction may be integrated as the sum of a polynomial and a certain number of partial fractions, i.e. it is also given by an elementary function, consisting of rational fractions, logarithms and arctangents.

The chief step in the integration of a rational fraction is its expansion into partial fractions, which requires a knowledge of the expansion of the denominator into real linear and quadratic factors (or, what amounts to the same thing, a knowledge of the zeros of the denominator). This last task is strictly speaking an algebraic rather than analytic problem, and we are not specially concerned with it here. It will be clear from the examples of Sec. 100 how the fraction is to be split into partial fractions, given a knowledge of the factors of the denominator.

A rational function $R(x)$ (only rational operations are performed on x : addition, subtraction, multiplication, division) can always be reduced to the form of a rational fraction. In view of this, the integration of rational functions leads in the long run to the extremely simple scheme described above.

100. Examples. The following examples will explain the general propositions of Sec. 99 on the integration of rational fractions.

(1) $\int \frac{x-3}{x^3-x} dx$. The expansion of the integrand into partial fractions must be as follows:

$$\frac{x-3}{x^3-x} = \frac{x-3}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

(obviously, the letters used to denote the as yet undetermined coefficients are of no significance).

We obtain on clearing fractions:

$$x-3 = A(x^2-1) + Bx(x+1) + Cx(x-1).$$

Since this is an identity, the coefficients of like powers of x must be equal:

$$0 = A + B + C, 1 = B - C, -3 = -A.$$

We find from this system of three equations with three unknowns:

$$A = 3, B = -1, C = -2.$$

Coefficients A, B, C may be found by an even simpler procedure: putting $x = 0, x = 1, x = -1$, we get $-3 = -A, -2 = 2B, -4 = 2C$, i.e. again

$$A = 3, B = -1, C = -2.$$

We thus have identically

$$\frac{x-3}{x^3-x} = \frac{3}{x} - \frac{1}{x-1} - \frac{2}{x+1};$$

hence

$$\begin{aligned} \int \frac{x-3}{x^3-x} dx &= 3 \int \frac{dx}{x} - \int \frac{dx}{x-1} - 2 \int \frac{dx}{x+1} \\ &= 3 \ln|x| - \ln|x-1| - 2 \ln|x+1| + C = \ln \frac{|x|^3}{|x-1|(x+1)^2} + C. \end{aligned}$$

The reader may recall that we have already integrated rational fractions by splitting them into partial fractions (Sec. 96, example 6). But there, instead of a general rule, we took as our basis directly obvious transformations, the use of which may in fact always be recommended in suitable cases.

(2) $\int \frac{x-5}{x^3-3x^2+4} dx$. We factorize the denominator: having observed that it vanishes for $x = -1$, we divide through by $x+1$; we obtain in the quotient $x^2 - 4x + 4 = (x-2)^2$ *, i.e. the expansion into partial fractions must have the form

$$\frac{x-5}{x^3-3x^2+4} = \frac{x-5}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{(x-2)^2} + \frac{C}{x-2}.$$

Let us find coefficients A, B, C . We have on clearing the fractions:

$$x-5 = A(x-2)^2 + B(x+1) + C(x+1)(x-2).$$

* The denominator may also be factorized directly here:

$$\begin{aligned} x^3 - 3x^2 + 4 &= x^3 + x^2 - 4x^2 + 4 = x^2(x+1) - 4(x-1)(x+1) \\ &= (x+1)(x^2 - 4x + 4) = (x+1)(x-2)^2. \end{aligned}$$

It is convenient here to put $x = -1$, $x = 2$, $x = 0$; we get

$$-6 = 9A, \quad -3 = 3B, \quad -5 = 4A + B - 2C.$$

Hence

$$A = \frac{2}{3}, \quad B = -1, \quad C = \frac{2}{3}.$$

Consequently,

$$\begin{aligned} \int \frac{x-5}{x^3-3x^2+4} dx &= -\frac{2}{3} \int \frac{dx}{x+1} - \int \frac{dx}{(x-2)^2} + \frac{2}{3} \int \frac{dx}{x-2} \\ &= -\frac{2}{3} \ln|x+1| + \frac{1}{x-2} + \frac{2}{3} \ln|x-2| + C \\ &= \frac{1}{x-2} + \frac{2}{3} \ln \left| \frac{x-2}{x+1} \right| + C. \end{aligned}$$

(3) $\int 12/(x^4 + x^3 - x - 1) dx$. The denominator is easily factorized:

$$\begin{aligned} x^4 + x^3 - x - 1 &= x^3(x+1) - (x+1) = (x+1)(x^3-1) \\ &= (x+1)(x-1)(x^2+x+1). \end{aligned}$$

We split the integrand into partial fractions:

$$\begin{aligned} \frac{12}{x^4 + x^3 - x - 1} &= \frac{12}{(x+1)(x-1)(x^2+x+1)} \\ &= \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1}. \end{aligned}$$

Coefficients A, B, C, D are obtained from the identity

$$\begin{aligned} 12 &= A(x-1)(x^2+x+1) + B(x+1)(x^2+x+1) + \\ &\quad + (Cx+D)(x+1)(x-1). \end{aligned}$$

On substituting four different values of x , say $x = 1$, $x = -1$, $x = 0$, $x = 2$, we get the system

$$\begin{aligned} 12 &= 6B, \quad 12 = -2A, \quad 12 = -A + B - D, \\ 12 &= 7A + 21B + 3(2C + D), \end{aligned}$$

from which we find:

$$A = -6, \quad B = 2, \quad C = 4, \quad D = -4.$$

Thus

$$\int \frac{12}{x^4 + x^3 - x - 1} dx = -6 \int \frac{dx}{x+1} + 2 \int \frac{dx}{x-1} + 4 \int \frac{x-1}{x^2+x+1} dx.$$

The last integral is evaluated thus:

$$\begin{aligned}
 \int \frac{x-1}{x^2+x+1} dx &= \frac{1}{2} \int \frac{2x+1-3}{x^2+x+1} dx \\
 &= \frac{1}{2} \int \frac{2x+1}{x^2+x-1} dx - \frac{3}{2} \int \frac{dx}{x^2+x+1} \\
 &= \frac{1}{2} \ln(x^2+x+1) - \frac{3}{2} \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} \\
 &= \frac{1}{2} \ln(x^2+x+1) - \frac{3}{2} \frac{1}{\sqrt{\frac{3}{4}}} \arctan \frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}} + C.
 \end{aligned}$$

We finally have:

$$\begin{aligned}
 \int \frac{x^2}{x^4+x^3-x-1} dx &= -6 \ln|x+1| + 2 \ln|x-1| + \\
 &\quad + 2 \ln(x^2+x+1) - 4\sqrt{3} \arctan \frac{2x+1}{\sqrt{3}} + C \\
 &= \ln \frac{(x^3-1)^2}{(x+1)^6} - 4\sqrt{3} \arctan \frac{2x+1}{\sqrt{3}} + C.
 \end{aligned}$$

(4) $\int \frac{3x^4+x^3+4x^2+1}{x^5+2x^3+x} dx$. The denominator obviously has the factors:

$$x^5+2x^3+x = x(x^4+2x^2+1) = x(x^2+1)^2.$$

The rational fraction must therefore be split into partial fractions as follows:

$$\begin{aligned}
 \frac{3x^4+x^3+4x^2+1}{x^5+2x^3+x} &= \frac{3x^4+x^3+4x^2+1}{x(x^2+1)^2} \\
 &= \frac{A}{x} + \frac{Bx+C}{(x^2+1)^2} + \frac{Dx+E}{x^2+1}.
 \end{aligned}$$

This gives us:

$$\begin{aligned}
 3x^4+x^3+4x^2+1 &= A(x^2+1)^2 + (Bx+C)x + (Dx+E)x(x^2+1),
 \end{aligned}$$

or, on collecting the like terms on the right-hand side:

$$\begin{aligned} 3x^4 + x^3 + 4x^2 + 1 &= \\ &= (A + D)x^4 + Ex^3 + (2A + B + D)x^2 + (C + E)x + A. \end{aligned}$$

On equating coefficients, we get the system

$$A + D = 3, \quad E = 1, \quad 2A + B + D = 4, \quad C + E = 0, \quad A = 1,$$

from which we find:

$$A = 1, \quad B = 0, \quad C = -1, \quad D = 2, \quad E = 1.$$

Consequently,

$$\begin{aligned} \int \frac{3x^4 + x^3 + 4x^2 + 1}{x^5 + 2x^3 + x} dx &= \int \frac{dx}{x} - \int \frac{dx}{(x^2 + 1)^2} + \int \frac{2x + 1}{x^2 + 1} dx \\ &= \int \frac{dx}{x} - \int \frac{x^2 + 1 - x^2}{(x^2 + 1)^2} dx + \int \frac{2x}{x^2 + 1} dx + \int \frac{dx}{x^2 + 1} \\ &= \ln|x| + \ln(x^2 + 1) + \int \frac{x^2}{(x^2 + 1)^2} dx. \end{aligned}$$

Further integration by parts gives us*:

$$\begin{aligned} \int x \frac{dx}{(x^2 + 1)^2} &= -\frac{1}{2} \frac{x}{x^2 + 1} + \frac{1}{2} \int \frac{dx}{1 + x^2} \\ &= -\frac{x}{2(x^2 + 1)} + \frac{1}{2} \arctan x + C \end{aligned}$$

and finally:

$$\begin{aligned} \int \frac{3x^4 + x^3 + 4x^2 + 1}{x^5 + 2x^3 + x} dx &= -\frac{x}{2(x^2 + 1)} + \\ &+ \ln|x(x^2 + 2)| + \frac{1}{2} \arctan x + C. \end{aligned}$$

REMARK. A commonly encountered type of rational fraction can be integrated much more easily than by using the general rule. It is

$$\int \frac{dx}{(x-a)^n (x-b)^m},$$

* The integral $\int [x^2/(x^2 + 1)^2] dx$ can also readily be found by the trigonometric substitution $x = \tan u$.

where n and m are positive integers; we use the substitution

$$\frac{x-a}{x-b} = z,$$

which reduces the integral at once to a sum of power functions. In fact:

$$x-a = z(x-b), \quad \text{i.e.} \quad (x-b) + (b-a) = z(x-b),$$

whence

$$x-b = \frac{b-a}{z-1}, \quad x-a = (b-a) \frac{z}{z-1}, \quad dx = -\frac{b-a}{(z-1)^2} dz.$$

Thus

$$\begin{aligned} \int \frac{dx}{(x-a)^n (x-b)^m} &= - \int \frac{(b-a)(z-1)^n (z-1)^m dz}{(b-a)^n z^n (b-1)^m (z-1)^2} \\ &= - \frac{1}{(b-a)^{m+n-1}} \int \frac{(z-1)^{m+n-2}}{z^n} dz. \end{aligned}$$

On expanding the binomial by Newton's formula, we get the integral of a sum of power functions.

101. Ostrogradskii's method. The integration of rational fractions leads to the longest working when finding the integrals of partial fractions where the denominator has multiple roots (cases 2 and 4 of Sec. 99). M. V. Ostrogradskii (see Sec. 4) proposed a method for finding the integral of a rational fraction which avoids the labour involved in the integration of partial fractions in cases two and four.

The method is as follows. Suppose we are given

$$\int \frac{P(x)}{Q(x)} dx,$$

where

$$Q(x) = (x-\alpha)^k \dots (x^2+px+q)^t.$$

To each factor $(x-\alpha)^k$ there correspond k partial fractions:

$$\frac{A_k}{(x-\alpha)^k} + \dots + \frac{A_2}{(x-\alpha)^2} + \frac{A_1}{x-\alpha},$$

the integration of which gives the sum of two terms: (1) the rational fraction $M(x)/(x-\alpha)^{k-1}$, where $M(x)$ is a polynomial of degree not exceeding $(k-2)$, and (2) the integral $\int \frac{A}{x-\alpha} dx$; to each factor $(x^2+px+q)^t$ there correspond t partial fractions:

$$\frac{B_t x + C_t}{(x^2+px+q)^t} + \dots + \frac{B_1 x + C_1}{(x^2+px+q)},$$

the integration of which yields the sum of two terms: (1) the rational fraction $N(x)/(x^2 + px + q)^{t-1}$, where $N(x)$ is a polynomial of degree not exceeding $2t - 3$, and (2) $\int \frac{Bx + C}{x^2 + px + q} dx$.

On summing the results corresponding to all the factors of the denominator $Q(x)$, we evidently get

$$\int \frac{P(x)}{Q(x)} dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)} dx, \quad (\dagger)$$

where

$$Q_1(x) = (x - \alpha)^{k-1} \cdots (x^2 + px + q)^{t-1},$$

$$Q_2(x) = (x - \alpha) \cdots (x^2 + px + q),$$

$P_1(x)$ is a polynomial of degree lower than the degree of $Q_1(x)$, and $P_2(x)$ is of degree lower than the degree of $Q_2(x)$.

Formula ($\dagger$) reveals the analytic structure of the integral of a rational fraction and is called Ostrogradskii's formula.

We see now that the fraction $P_1(x)/Q_1(x)$ —called the rational part of the integral—can be found without any integration, by means of purely algebraic transformations; thus Ostrogradskii's formula shows that the integration of any rational fraction in fact reduces to integration of some other rational fraction whose denominator has only simple (single) zeroes, coinciding with the various zeros of the denominator of the given fraction.

If the factors of $Q(x)$ are known, the formation of polynomials $Q_1(x)$ and $Q_2(x)$ presents no difficulty; here, $Q_1(x)Q_2(x) = Q(x)$. We write down with undetermined coefficients the polynomials: $P_1(x)$ of degree $n_1 - 1$, where n_1 is the degree of $Q_1(x)$, $P_2(x)$ of degree $n_2 - 1$, where n_2 is the degree of $Q_2(x)$ ($n_1 + n_2 = n$, where n is the degree of $Q(x)$). To find these coefficients, we take the derivatives of both sides of equation ($\dagger$):

$$\frac{P(x)}{Q(x)} = \frac{P_1'(x)Q_1(x) - Q_1'(x)P_1(x)}{Q_1^2(x)} + \frac{P_2(x)}{Q_2(x)}. \quad (\dagger\dagger)$$

This formula, which is equivalent to Ostrogradskii's, is an identity. On clearing fractions and equating coefficients of like powers of x on the left and right-hand sides, we get a system of n equations for the $n_1 + n_2 (= n)$ unknown coefficients of polynomials $P_1(x)$ and $P_2(x)$. (Of course the system can be got by substituting n different numerical values for x in ($\dagger\dagger$) instead of by equating coefficients; see Sec. 100.)

The polynomial $Q_1(x)$ (and then $Q_2(x) \frac{Q(x)}{Q_1(x)}$) can easily be found even when the factors of the denominator $Q(x)$ are unknown; $Q_1(x)$ is in fact the highest common factor of $Q(x)$ and $Q'(x)$. For, as we know, a k -ple zero of the function $Q(x)$ is a $(k-1)$ -ple zero of its derivative $Q'(x)$; hence each "multiple" factor in the expansion of polynomial $Q(x)$ also leads to a factor in the expansion of $Q'(x)$, but reduced in degree by unity. Thus

$$Q'(x) = (x - \alpha)^{k-1} \dots (x^2 + px + q)^{k-1} \cdot Q_3(x)$$

where the polynomial $Q_3(x)$ has no zeros in common with $Q(x)$. It is clear from this form that $Q_1(x)$ is the highest common factor of the polynomials

$$Q(x) = Q_1(x)Q_2(x)$$

$$Q'(x) = Q_1(x)Q_3(x).$$

As with every highest common factor of two polynomials (or numbers), $Q_1(x)$ can be found with the aid of the method of successive division (called Euclid's algorithm). In other words, $Q(x)$ is divided by $Q'(x)$, then $Q'(x)$ is divided by the remainder, then the first remainder by the second, and so on, until the division leaves no remainder. The last remainder thus obtained is in fact the highest common factor. Constant factors of both dividend and divisor can be removed and combined, since this only affects the constant factor of the final result, which we can always take into the polynomial $P_1(x)$.

It will be clear from an example how Ostrogradskii's method simplifies the integration of rational fractions even in the most difficult cases.

Example. We take example (4) of the previous section:

$$I = \int \frac{3x^4 + x^3 + 4x^2 + 1}{x^5 + 2x^3 + x} dx.$$

Here $Q(x) = x^5 + 2x^3 + x$, $Q'(x) = 5x^4 + 6x^2 + 1$. We find the highest common factor of $Q(x)$ and $Q'(x)$. We divide $Q(x)$ by $Q'(x)$, then $Q'(x)$ by the new remainder, and so on:

$$\begin{array}{r|l} 5x^5 + 10x^3 + 5x & 5x^4 + 6x^2 + 1 \\ \hline 5x^5 + 6x^3 + x & x \mid 5x^4 + 5x^2 \quad 5x \mid x^3 + x \quad x \mid \\ \hline 4x^3 + 4x & \quad \quad \quad x^2 + 1 \quad \quad \quad = \end{array}$$

Thus the highest common factor is $x^2 + 1$, i.e. $Q_1(x) = x^2 + 1$, i.e.

$$Q_2(x) = \frac{Q(x)}{Q_1(x)} = \frac{x^5 + 2x^3 + x}{x^2 + 1} = x^3 + x = x(x^2 + 1).$$

We have by Ostrogradskii's formula:

$$I = \int \frac{3x^4 + x^3 + 4x^2 + 1}{x^5 + 2x^3 + x} dx = \frac{Ax + B}{x^2 + 1} + \int \frac{Cx^2 + Dx + E}{x(x^2 + 1)} dx,$$

whence

$$\frac{3x^4 + x^3 + 4x^2 + 1}{x^5 + 2x^3 + x} = \frac{Ax^2 + A - 2Ax^2 - 2Bx}{(x^2 + 1)^2} + \frac{Cx^2 + Dx + E}{x(x^2 + 1)}.$$

On bringing to a common denominator and neglecting it, we get:

$$3x^4 + x^3 + 4x^2 + 1 = (-Ax^2 - 2Bx + A)x + (Cx^2 + Dx + E)(x^2 + 1).$$

Comparison of the coefficients of like powers gives a system of five equations with five unknowns:

$$\begin{aligned} 3 &= C, & 0 &= A + D, \\ 1 &= -A + D, & 1 &= E, \\ 4 &= -2B + C + E, \end{aligned}$$

whence

$$A = -\frac{1}{2}, \quad B = 0, \quad C = 3, \quad D = \frac{1}{2}, \quad E = 1.$$

Consequently,

$$I = -\frac{x}{2(x^2+1)} + \frac{1}{2} \int \frac{6x^2+x+2}{x^3+x} dx,$$

and simple integration of the fraction whose denominator has only simple roots $(0, +i, -i)$ leads to the final answer:

$$I = -\frac{x}{2(x^2+1)} + \ln|x(x^2+1)| + \frac{1}{2} \arctan x + C.$$

102. Some irrational functions. We now turn to the integration of some simple types of irrational function.

The main efforts are most often directed to finding transformations which reduce the given integral to the integral of a rational function. Such transformations are said to rationalize the integral.

Note that the symbol $R(\alpha, \beta, \gamma, \dots)$ is used to denote an expression rational with respect to $\alpha, \beta, \gamma, \dots$, i.e. such that only rational operations are performed on $\alpha, \beta, \gamma, \dots$

I. Suppose that the integrand is a rational expression in x (the variable of integration) and in various radicals of the same rational linear (in particular, linear) function $\frac{ax+b}{a_1x+b_1}$ (on the assumption that $ab_1 - ba_1 \neq 0$, see Sec. 19):

$$y = R\left(x, \sqrt[m]{\frac{ax+b}{a_1x+b_1}}, \sqrt[p]{\frac{ax+b}{a_1x+b_1}}, \dots\right).$$

If n is the least multiple of all the indices $m, p, \dots$, by using the substitution

$$\frac{ax+b}{a_1x+b_1} = u^n,$$

the integral in question can be reduced (rationalized) to an integral of a rational fraction, i.e. expressed by an elementary function.

For we have:

$$x = \frac{b_1 u^n - b}{a - a_1 u^n}, \quad dx = \frac{ab_1 - ba_1}{(a - a_1 u^n)^2} \cdot n u^{n-1} du$$

and

$$\begin{aligned} & \int R\left(x, \sqrt[m]{\frac{ax+b}{a_1x+b_1}}, \sqrt[p]{\frac{ax+b}{a_1x+b_1}}, \dots\right) dx \\ &= n(a_1b - ba_1) \int R\left(\frac{b_1u^n - b}{a - a_1u^n}, u^{n/m}, u^{n/p}, \dots\right) \frac{u^{n-1}}{(a - a_1u^n)^2} du. \end{aligned}$$

It is clear from this that the integrand is rational in the variable of integration u^* ($n/m, n/p, \dots$ are integers).

Example. $\int dx/(x - \sqrt[3]{3x+2})$. We put $3x+2 = u^3$. Then

$$x = \frac{u^3 - 2}{3}, \quad dx = u^2 du.$$

We get:

$$\int \frac{dx}{x - \sqrt[3]{3x+2}} = \int \frac{u^2}{\frac{u^3-2}{3} - u} du = 3 \int \frac{u^2}{u^3 - 3u - 2} du.$$

It remains to apply the familiar method of integrating rational fractions to the last integral. We obtain:

$$\begin{aligned} \int \frac{3u^2}{u^3 - 3u - 2} du &= - \int \frac{du}{(u+1)^2} + \frac{5}{3} \int \frac{du}{u+1} + \frac{4}{3} \int \frac{du}{u-2} \\ &= \frac{1}{u+1} + \frac{5}{3} \ln|u+1| + \frac{4}{3} \ln|u-2| + C \end{aligned}$$

and, on returning to the original variable:

$$\begin{aligned} \int \frac{dx}{x - \sqrt[3]{3x+2}} &= \frac{1}{\sqrt[3]{3x+2} + 1} + \frac{5}{3} \ln|\sqrt[3]{3x+2} + 1| + \\ &+ \frac{4}{3} \ln|\sqrt[3]{3x+2} - 2| + C. \end{aligned}$$

II. The expression

$$x^m(a + bx^n)^p dx$$

(a and b are real numbers, m, n and p are rational numbers) is called a differential binomial. We shall assume that m and n are integers; otherwise, the substitution $x = u^q$ is used, where q is the common denominator of fractions m and n , so that we arrive at the same form of binomial. Let $p = r/s$, $s > 0$. If p is a positive integer,

* A rational function of a rational function is again rational.

we simply have a polynomial; but if it is a negative integer, we have a linear rational function.

Two substitutions can be used for integration of a differential binomial when p is not an integer:

$$(1) \quad a + b x^n = u^s.$$

$$(2) \quad a x^{-n} + b = u^s.$$

The first is used when $(m+1)/n$ is an integer, the second when $(m+1)/n + p$ is an integer. The integral is here transformed to an integral of a linear rational function (is rationalized), i.e. is expressed in terms of elementary functions.

The proof of this assertion is extremely simple, and the reader can carry it out for himself. P. L. Chebyshev (see Sec. 4) showed that the integration in elementary functions of the differential binomial is only possible in the following three cases:

(1) p an integer; (2) $(m+1)/n$ an integer; (3) $(m+1)/n + p$ an integer.

Examples. (1) $\int \frac{\sqrt{1+x^2}}{x} dx$. Here, $(m+1)/n = 0$. We therefore put $1+x^2 = u^2$. Hence $x dx = u du$ and

$$\begin{aligned} \int \frac{\sqrt{1+x^2}}{x} dx &= \int x \frac{\sqrt{1+x^2}}{x^2} dx = \int \frac{u}{u^2-1} u du \\ &= \int du + \int \frac{du}{u^2-1} = u + \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| + C \\ &= \sqrt{1+x^2} + \frac{1}{2} \ln \frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1} + C. \end{aligned}$$

(2) $\int dx/x^2 (2+x^3)^{5/3}$. Here $\frac{m+1}{n} = -\frac{1}{3}$, whilst $\frac{m+1}{n} + p = -2$.

We put

$$2x^{-3} + 1 = u^3.$$

Hence

$$x^3 = \frac{2}{u^3-1}, \quad x^2 dx = -\frac{2u^2}{(u^3-1)^2} du$$

and

$$\begin{aligned}\int \frac{dx}{x^2(2+x^3)^{1/2}} &= \int \frac{x^2 dx}{x^4(2+x^3)^{1/2}} = - \int \frac{2u^2 du}{(u^3-1)^2 \frac{2^3}{(u^3-1)^3} u^5} \\ &= -\frac{1}{4} \int \frac{u^3-1}{u^3} du = -\frac{1}{4} \int du + \frac{1}{4} \int u^{-3} du \\ &= -\frac{1}{4} u - \frac{1}{8} \frac{1}{u^2} + C = -\frac{1}{8} \frac{4+3x^3}{x(2+x^3)^{1/2}} + C.\end{aligned}$$

III. We consider some frequently encountered integrals depending on the irrational expression $\sqrt{ax^2+bx+c}$.

(1) $\int dx/\sqrt{ax^2+bx+c}$. We take $1/\sqrt{a}$ (if $a > 0$) outside the integral sign (or $1/\sqrt{-a}$ (if $a < 0$)) and reduce the integral to the form

$$\int \frac{dx}{\sqrt{x^2+px+q}} \quad \text{or} \quad \int \frac{dx}{\sqrt{-x^2+px+q}}.$$

On completing the squares, we get

$$\int \frac{dx}{\sqrt{\left(x+\frac{p}{2}\right)^2 + \frac{4q-p^2}{4}}} \quad \text{or} \quad \int \frac{dx}{\sqrt{\frac{4q+p^2}{4} - \left(x-\frac{p}{2}\right)^2}}.$$

The first integral is expressed by a logarithm (Sec. 98, example 6), and the second by arcsin (Sec. 98, example 5) if $4q+p^2 > 0$; whereas if $4q+p^2 < 0$, the integrand only takes imaginary values.

(2) $\int dx/(x-\alpha)\sqrt{ax^2+bx+c}$. It is easily shown that the substitution $x-\alpha = 1/u$ reduces this integral to an integral of form (1).

(3) $\int \frac{Ax+B}{\sqrt{ax^2+bx+c}} dx$. This integral can be split into two by separating out the part of the numerator proportional to the derivative of the expression under the root sign; the integral of this part is now got directly as the integral of a power function, whilst the second is an integral of form (1).

We shall give without proof a simple scheme of integration when the numerator is any polynomial. In fact:

$$\int \frac{P(x)}{\sqrt{ax^2+bx+c}} dx = P_1(x) \sqrt{ax^2+bx+c} + \lambda \int \frac{dx}{\sqrt{ax^2+bx+c}} \quad (\dagger)$$

where $P_1(x)$ is a polynomial of degree equal to the degree minus one of $P(x)$, and λ is a constant. We find $P_1(x)$ and λ by differentiating the above equation:

$$\frac{P(x)}{\sqrt{ax^2 + bx + c}} = P_1'(x)\sqrt{ax^2 + bx + c} + \frac{P_1(x)(2ax + b)}{2\sqrt{ax^2 + bx + c}} + \frac{\lambda}{\sqrt{ax^2 + bx + c}},$$

whence

$$2P(x) = 2P_1'(x)(ax^2 + bx + c) + P_1(x)(2ax + b) + 2\lambda.$$

We can find from this identity all the coefficients of polynomial $P_1(x)$ and λ . The integration is completed by finding an integral of type (1).

(4) $\int \sqrt{ax^2 + bx + c} dx$. By completing the square in the expression under the root sign, this is reduced to one of the three familiar integrals (Sec. 98, examples 5 and 8):

$$\int \sqrt{1 - x^2} dx, \quad \int \sqrt{x^2 \pm 1} dx.$$

Alternatively, we can write

$$\int \sqrt{ax^2 + bx + c} dx = \int \frac{ax^2 + bx + c}{\sqrt{ax^2 + bx + c}} dx;$$

the last integral is found in accordance with scheme(†). The integral

$$\int P(x) \sqrt{ax^2 + bx + c} dx,$$

where $P(x)$ is any polynomial, can be found in a similar way.

We shall consider in Sec. 104 a general method for integrating expressions rationally dependent on x and $\sqrt{ax^2 + bx + c}$.

103. Trigonometric functions. Suppose we have an expression which is rationally dependent on trigonometric functions. Since all the trigonometric functions are expressible rationally in terms of $\sin x$ and $\cos x$, our expressions can be looked on as a rational function of $\sin x$ and $\cos x$, i.e. it has the form $R(\sin x, \cos x)$.

THEOREM. The integral

$$\int R(\sin x, \cos x) dx$$

is transformed by the substitution $u = \tan \frac{1}{2}x$ to an integral of a rational function (i.e. is rationalized).

Proof. We have:

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}},$$

i.e.

$$\sin x = \frac{2u}{1 + u^2}.$$

Further,

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \frac{1}{1 + \tan^2 \frac{x}{2}} - \frac{\tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}},$$

i.e.

$$\cos x = \frac{1 - u^2}{1 + u^2}.$$

Finally, we have from the equation $x = 2 \arctan u$:

$$dx = \frac{2}{1 + u^2} du.$$

Therefore

$$\int R(\sin x, \cos x) dx = \int R\left(\frac{2u}{1 + u^2}, \frac{1 - u^2}{1 + u^2}\right) \frac{2}{1 + u^2} du.$$

The integrand here is rational in u .

Example. $\int dx/(3 + 5 \cos x)$. We put $u = \tan \frac{1}{2}x$. Then

$$\begin{aligned} \int \frac{dx}{3 + 5 \cos x} &= \int \frac{2 du}{(1 + u^2) \left(3 + 5 \frac{1 - u^2}{1 + u^2}\right)} = \int \frac{du}{4 - u^2} \\ &= \frac{1}{4} \ln \left| \frac{2 + u}{2 - u} \right| + C = \frac{1}{4} \ln \left| \frac{2 + \tan \frac{x}{2}}{2 - \tan \frac{x}{2}} \right| + C. \end{aligned}$$

We made use earlier (Sec. 96, example 9 and Sec. 98, example 6) of transformation with the aid of the tangent of half the argument.

This method of integrating a rational trigonometric differential always gives results, though, precisely because of its generality, it is often not the best transformation from the point of view of brevity and simplicity.

For instance, if $\sin x$ and $\cos x$ only appear with even powers in the expression for the function R , the substitution $u = \tan x$ is more convenient. We get:

$$\int R(\sin^2 x, \cos^2 x) dx = \int R\left(\frac{u^2}{1+u^2}, \frac{1}{1+u^2}\right) \frac{1}{1+u^2} du.$$

For example:

$$\begin{aligned} \int \frac{dx}{1+3\cos^2 x} &= \int \frac{du}{(1+u^2)\left(1+\frac{3}{1+u^2}\right)} = \int \frac{du}{4+u^2} \\ &= \frac{1}{2} \arctan \frac{u}{2} + C = \frac{1}{2} \arctan \frac{\tan x}{2} + C. \end{aligned}$$

The general substitution $u = \tan \frac{1}{2}x$ can obviously also be used here, but it leads to lengthier working.

The substitution $u = \tan x$ is also easy to use in an integral such as

$$\int \frac{dx}{1+2\tan x}.$$

We have:

$$\int \frac{dx}{1+2\tan x} = \int \frac{du}{(1+u^2)(1+2u)}.$$

On finding the integral of the rational fraction and returning to the old variable, we get:

$$\int \frac{dx}{1+2\tan x} = \frac{1}{5} [x + 2 \ln |\cos x + 2 \sin x|] + C.$$

It is also often convenient to put $u = \sin x$ or $u = \cos x$. These substitutions were employed in Sec. 96.

The shortest way to the final result is sometimes by a method other than a substitution method.

To illustrate, let us take integrals of the form

$$\int \sin^n x \cos^m x dx,$$

where n and m are integers. If one of them is odd, the simplest thing is to put $u = \sin x$ or $u = \cos x$.

For instance ($n = -3$, $m = -1$):

$$\int \frac{dx}{\sin^3 x \cos x} = \int \frac{\cos x}{\sin^3 x \cos^2 x} dx = \int \frac{du}{u^3(1-u^2)}, \quad u = \sin x,$$

and it remains to find the integral of a rational fraction. We get

$$\begin{aligned} \int \frac{du}{u^3(1-u^2)} &= \int \frac{du}{u^3} + \int \frac{du}{u} + \frac{1}{2} \int \frac{du}{1-u} - \frac{1}{2} \int \frac{du}{1+u} \\ &= -\frac{1}{2u^2} + \ln \left| \frac{u}{1-u^2} \right| + C = -\frac{1}{2\sin^2 x} + \ln |\tan x| + C. \end{aligned}$$

When n and m are both even, we can put $u = \tan x$.

For instance ($n = -4$, $m = 0$):

$$\int \frac{dx}{\sin^4 x} = \int \frac{dx}{\cos^2 x \sin^2 x \tan^2 x} = \int \frac{du}{\frac{u^2}{1+u^2} u^2}, \quad u = \tan x.$$

Consequently,

$$\int \frac{dx}{\sin^4 x} = \int \frac{1+u^2}{u^4} du = -\frac{1}{3u^3} - \frac{1}{u} + C = -\frac{1}{3\tan^3 x} - \frac{1}{\tan x} + C.$$

But if both powers are positive as well, the trigonometric transformations to multiple arguments can be used successfully.

For instance:

$$\begin{aligned} \int \sin^2 x \cos^4 x dx &= \int (\sin x \cos x)^2 \cos^2 x dx \\ &= \frac{1}{8} \int \sin^2 2x (1 + \cos 2x) dx \\ &= \frac{1}{8} \int \sin^2 2x dx + \frac{1}{8} \int \sin^2 2x \cos 2x dx \\ &= \frac{1}{64} \int (1 - \cos 4x) d(4x) + \frac{1}{16} \int \sin^2 2x d(\sin 2x) \\ &= \frac{1}{16} x - \frac{1}{64} \sin 4x + \frac{1}{48} \sin^3 2x + C. \end{aligned}$$

With integral and non-negative n and m (even or odd), the method of integration by parts is extremely convenient: here it gives a reduction (recurrence) formula. We encountered examples of reduction formulae when integrating rational fractions in Sec. 99.

Let us find

$$I_m = \int \cos^m x \, dx, \quad n = 0, \quad m > 0.$$

We put

$$u = \cos^{m-1} x, \quad dv = \cos x \, dx.$$

Now:

$$du = -(m-1) \cos^{m-2} x \sin x \, dx, \quad v = \sin x$$

and

$$\begin{aligned} I_m &= \sin x \cos^{m-1} x + (m-1) \int \cos^{m-2} x \sin^2 x \, dx \\ &= \sin x \cos^{m-1} x + (m-1) \int \cos^{m-2} x \, dx - (m-1) \int \cos^m x \, dx. \end{aligned}$$

whence

$$m I_m = \sin x \cos^{m-1} x + (m-1) I_{m-2}.$$

so that

$$I_m = \frac{1}{m} \sin x \cos^{m-1} x + \frac{m-1}{m} I_{m-2}. \quad (*)$$

This is the reduction formula which we wanted. The integration problem can easily be completed by this means.

Similar reduction formulae may be found for the cases $n > 0$, $m = 0$ and $n > 0$, $m > 0$.

Notice the difference in these three methods, which enable the integration of rational trigonometric differentials to be simplified in cases where the general method (i.e. the substitution $u = \tan \frac{1}{2} x$) is very complicated.

104. Rational functions of x and $\sqrt{ax^2 + bx + c}$. We now investigate in the general form the integration of functions rationally dependent on x and the irrational $\sqrt{ax^2 + bx + c}$, i.e. functions of the form $R(x; \sqrt{ax^2 + bx + c})$.

Important particular cases have been considered in Sec. 102, III. There is a general method, however, for rationalizing integrals

$$\int R(x, \sqrt{ax^2 + bx + c}) \, dx \quad (\dagger)$$

i.e. for integration of them in terms of elementary functions. The method is based on the following scheme.

First of all, by separating out the complete square in the expression under the root sign, and using a linear substitution, we reduce the integral to one of the three types:

$$\int R(y, \sqrt{1-y^2}) \, dy, \quad (\text{A})$$

$$\int R(y, \sqrt{y^2-1}) \, dy, \quad (\text{B})$$

$$\int R(y, \sqrt{y^2+1}) \, dy. \quad (\text{C})$$

Then, by substituting $y = \sin u$, $y = 1/\sin u$, $y = \tan u$, integrals (A), (B), (C) are transformed respectively to integrals of rational trigonometric differentials:

$$\begin{aligned}\int R(y, \sqrt{1-y^2}) dy &= \int R(\sin u, \cos u) \cos u \, du, \\ \int R(y, \sqrt{y^2-1}) dy &= -\int R\left(\frac{1}{\sin u}, \frac{\cos u}{\sin u}\right) \frac{\cos u}{\sin^2 u} \, du, \\ \int R(y, \sqrt{y^2+1}) dy &= \int R\left(\frac{\sin u}{\cos u}, \frac{1}{\cos u}\right) \frac{1}{\cos^2 u} \, du.\end{aligned}$$

Finally, these last integrals are known (Sec. 103) to be rationalizable (say with the general substitution $\tan \frac{1}{2}u = z$).

The reduction of integrals (A), (B), (C) to integrals of rational functions can be achieved in one step instead of two. We only need to express y directly in terms of z , passing over the intermediate variable u . We have:

$$\begin{aligned}y = \sin u &= \frac{2z}{1+z^2} && \text{in case (A),} \\ y = \frac{1}{\sin u} &= \frac{1+z^2}{2z} && \text{in case (B),} \\ y = \tan u &= \frac{2z}{1-z^2} && \text{in case (C).}\end{aligned}$$

These formulae in fact give the substitutions which directly rationalize integrals (A), (B) and (C). This means that we can easily lay down formulae for substitutions which rationalize the original integral of

$$R(x, \sqrt{ax^2 + bx + c})$$

in different cases (called Euler's substitutions), though there is no need for these in practice.

It is sometimes preferable not to carry through the scheme to its conclusion, but to change over to the transformation of the integrals concerned to integrals of trigonometric functions.

The latter—as will be seen in the next section—can in fact often be found more simply by methods other than reduction, in accordance with the general rule, to integrals of a rational function.

For instance, the integral

$$\int \frac{x^n}{(1+x^2)^m} dx,$$

where m and n are integers, is transformed by the substitution $x = \tan u$ to

$$\int \sin^n u \cos^{m-n-2} u \, du,$$

which it would be pointless further to transform to an integral of a rational function. It is more conveniently found by special methods (e.g. with the aid of reduction formulae when $m \geq n + 2$). Thus direct rationalization of the given integral in accordance with the above formula would be unnecessarily complicated.

Examples. (1) $\int dx/(\sqrt{1-x^2}-1)$. We put $x = \sin u$, $dx = \cos u \, du$. Then

$$\int \frac{dx}{\sqrt{1-x^2}-1} = \int \frac{\cos u}{\cos u - 1} \, du = u - \int \frac{du}{1 - \cos u}$$

and further,

$$\begin{aligned} \int \frac{du}{1 - \cos u} &= \int \frac{1 + \cos u}{\sin^2 u} \, du = \int \frac{du}{\sin^2 u} + \int \frac{d(\sin u)}{\sin^2 u} \\ &= -\cot u - \frac{1}{\sin u} + C. \end{aligned}$$

Thus

$$\begin{aligned} \int \frac{dx}{\sqrt{1-x^2}-1} &= u + \cot u + \frac{1}{\sin u} + C \\ &= \arcsin x + \frac{\sqrt{1-x^2}}{x} + \frac{1}{x} + C = \arcsin x + \frac{1 + \sqrt{1-x^2}}{x} + C. \end{aligned}$$

(2) $\int \frac{x + \sqrt{x^2 - 2x + 5}}{(x-1)\sqrt{x^2 - 2x + 5}} \, dx$. On completing the square in the expression under the root sign and putting $\frac{1}{2}(x-1) = y$, the integral becomes

$$\int \frac{y + \frac{1}{2} + \sqrt{y^2 + 1}}{y \sqrt{y^2 + 1}} \, dy.$$

Direct rationalization in accordance with the formula $y = 2a/(1-z^2)$ gives

$$\frac{1}{2} \int \frac{3+z}{z(1-z)} \, dz = \frac{3}{2} |z| - 2 \ln |1-z| + C.$$

We find on returning to the original variable:

$$\int \frac{x + \sqrt{x^2 - 2x + 5}}{(x-1)\sqrt{x^2 - 2x + 5}} dx = \frac{3}{2} \ln |-2 \pm \sqrt{x^2 - 2x + 5}| - \\ - 2 \ln |x + 1 \pm \sqrt{x^2 - 2x + 5}| + \frac{1}{2} \ln |x - 1| + C.$$

When solving concrete problems, the general method can usually be replaced by particular methods. For instance, integral (2) is better found by splitting it into two instead of rationalizing:

$$\int \frac{dx}{x-1} + \int \frac{x}{(x-1)\sqrt{x^2 - 2x + 5}} dx.$$

The first is found at once, and the second by the method of Sec. 102.

At the same time, the general scheme of rationalization of integral(†) is of theoretical interest in that it establishes the possibility in principle of expressing any such integral by elementary functions.

105. General remarks.

I. THE TECHNIQUE OF INTEGRATION. We have now concluded our consideration of the basic classes of integrable functions and of the general techniques of integration.

It will have been seen that it is only possible to give a rule for integrating functions in a few cases. Even when there is a theoretical scheme for obtaining the final result, this is not always the best or most economic method. There are very often several methods of carrying out a given integration. The particular circumstances ought to suggest in each individual case the device whereby the final answer is arrived at most rapidly. A mastery of the operation of integration includes not merely a knowledge of how a given integral can be found in the long run, but also the skill in obtaining it with the minimum of time and effort.

We shall supplement the examples given above with two further simple integrals to illustrate what has been said.

Let us take

$$\int \frac{x^3}{(x+1)^4} dx.$$

It would be possible in this example to apply the general rule (splitting the integrand into partial fractions). But the preliminary

substitution $x + 1 = u$ is easily seen to simplify the problem considerably:

$$\int \frac{x^3}{(x+1)^4} dx = \int \frac{(u-1)^3}{u^4} du = \int \frac{du}{u} - 3 \int \frac{du}{u^2} + 3 \int \frac{du}{u^3} - \int \frac{du}{u^4}.$$

It should also be surmised that

$$\int \frac{dx}{1 + \sin 2x}$$

ought not to be found by the general method (substituting $u = \tan x$), because

$$\begin{aligned} \int \frac{dx}{1 + \sin 2x} &= \int \frac{dx}{1 + \cos\left(\frac{\pi}{2} - 2x\right)} = \frac{1}{2} \int \frac{dx}{\cos^2\left(\frac{\pi}{4} - x\right)} \\ &= -\frac{1}{2} \int \frac{d\left(\frac{\pi}{4} - x\right)}{\cos^2\left(\frac{\pi}{4} - x\right)}; \end{aligned}$$

the last integral is tabulated.

Ingenuity and skill are gained by practice in the solution of plenty of examples.

For practical purposes, use can often be made of various reference books and ready-made tables of commonly occurring integrals*.

II. Integration in terms of elementary functions. As distinct from differentiation, integration is not (as already remarked) an automatic operation which always enables an elementary function to be found as the primitive of the given elementary function. It can be shown in fact that no such elementary expression for the primitive exists in many cases. In other words, elementary functions can be quoted whose integrals are not expressed by any finite combinations of basic elementary functions. Such functions are said to be non-integrable in elementary functions (or non-integrable in a finite form). We may recall the non-integrability in the finite form, proved by Chebyshev (Sec. 103, II), of the function $x^m(a + bx^n)^p$, if none of the three numbers p , $(m+1)/n$ or $(m+1)/n + p$ is an integer.

The following are further examples of integrals which cannot be found in a finite form: $\int dx/\sqrt{P(x)}$, where $P(x)$ is any polynomial of degree exceeding two; $\int \frac{\sin x}{x} dx$, $\int \frac{\cos x}{x} dx$, $\int dx/\ln x$.

* See for example I. M. RYZHIK and I. S. GRADSHTEIN, *Tables of integrals, sums, series and products* (*Tablitsy integralov, summ, ryadov i proizvedenii*), Gost., 1952.

The integrands here are very simply constructed, yet it still happens that their integrals cannot be represented by any elementary functions.

A distinction must be made between the existence of a function and the representation of it with the aid of certain given methods, as for example with the aid of elementary functions. The integrals mentioned exist, but our means—all the basic elementary functions—prove insufficient to form finite expressions from them for the integrals.

The point is this. Suppose we allow ourselves to use all the basic elementary functions apart from the logarithmic. Then many familiar integrals become “undiscoverable”. For instance, it will now be impossible to express in the finite form $\int dx/x$, $\int dx/(x^2 - 1)$ etc.

For there is now no function among those available (which do not include the logarithmic) such that its derivative is $1/x$ or $1/(x^2 - 1)$ etc. Since the logarithmic function has in fact been included in our treatment, we are able to express these integrals; but there is nothing surprising in the fact that other integrals still remain “undiscoverable” in a finite form. To make them “discoverable”, we have to widen the class of basic functions which we agree to use. This is in fact done in analysis. Among the “undiscoverable” integrals, the particularly simple and important ones are distinguished, and a detailed study is made of the functions represented by them. These new functions complete our stock of means and make accessible to integration in a finite form a number of functions which are non-integrable in the old sense.

CHAPTER VII

METHODS

OF EVALUATING DEFINITE INTEGRALS. IMPROPER INTEGRALS

1. Methods of Evaluating Integrals

106. Definite integration by parts. In view of the fact that $\int_a^x f(x) dx$ is a primitive of $f(x)$, we can write:

$$\int f(x) dx = \int_a^x f(x) dx + C.$$

On the other hand, we have by the Newton-Leibniz formula:

$$\int_a^b f(x) dx = \int f(x) dx \Big|_a^b.$$

The exact nature of the connection between the concepts of definite and indefinite integrals is revealed by these two relationships.

The Newton-Leibniz formula shows that we now have a satisfactory means of evaluating definite integrals—indefinite integration. As a matter of fact, however, the rules for indefinite integration can be applied directly to definite integrals, with the result that the working is shortened considerably.

RULE FOR INTEGRATION BY PARTS:

$$\int_{x_1}^{x_2} u dv = uv \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} v du.$$

Proof. We have:

$$\int_{x_1}^{x_2} u dv = \int u dv \Big|_{x_1}^{x_2} = (uv - \int v du) \Big|_{x_1}^{x_2},$$

whence the required formula follows directly.

Instead of carrying to the end the indefinite integration by parts then making a double substitution, direct use can be made of this last formula.

Let us take as an example

$$I_n = \int_0^{\pi/2} \sin^n x \, dx.$$

We put

$$u = \sin^{n-1} x, \quad dv = \sin x \, dx.$$

Then

$$du = (n-1) \sin^{n-2} x \cos x \, dx, \quad v = -\cos x$$

and

$$I_n = -\cos x \sin^{n-1} x \Big|_0^{\pi/2} + (n-1) \int_0^{\pi/2} \sin^{n-2} x \cos^2 x \, dx.$$

The first term on the right-hand side is zero; on replacing $\cos^2 x$ by $1 - \sin^2 x$ in the second term, we get

$$I_n = (n-1) \int_0^{\pi/2} \sin^{n-2} x \, dx - (n-1) \int_0^{\pi/2} \sin^n x \, dx,$$

i.e.

$$I_n = (n-1) I_{n-2} - (n-1) I_n,$$

whence

$$I_n = \frac{n-1}{n} I_{n-2}. \quad (\dagger)$$

This equation gives a reduction (recurrence) formula. On replacing n in it by $n-2$, we get:

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$

i.e.

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} I_{n-4}.$$

On continuing to lower the index in this manner, we arrive at I_1 in the case of integral n with n odd: $n = 2m + 1$, or at I_0 if n is even: $n = 2m$.

Thus:

$$I_{2m+1} = \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdot \frac{2m-4}{2m-3} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot I_1,$$

$$I_{2m} = \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdot \frac{2m-5}{2m-4} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot I_0.$$

But

$$I_1 = \int_0^{\pi/2} \sin x \, dx = 1, \quad I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2}.$$

Consequently,

$$I_{2m+1} = \int_0^{\pi/2} \sin^{2m+1} x \, dx = \frac{2 \cdot 4 \cdot 6 \dots (2m-2) \cdot 2m}{3 \cdot 5 \cdot 7 \dots (2m-1) (2m+1)},$$

$$I_{2m} = \int_0^{\pi/2} \sin^{2m} x \, dx = \frac{1 \cdot 3 \cdot 5 \dots (2m-3) (2m-1)}{2 \cdot 4 \cdot 6 \dots (2m-2) \cdot 2m} \cdot \frac{\pi}{2},$$

where m is a positive integer.

More laborious working is needed if we seek as a preliminary the indefinite integral of $\sin^n x$.

107. Change of variable in a definite integral.

I. RULE FOR CHANGE OF VARIABLE (SUBSTITUTION). If a function $x = \psi(u)$ is continuous together with its derivative $\psi'(u)$ in an interval $[u_1, u_2]$, and if $x = \psi(u)$ does not go outside the interval of continuity of the function $f(x)^*$ when u varies in $[u_1, u_2]$, where $\psi(u_1) = x_1$, $\psi(u_2) = x_2$, then

$$\int_{x_1}^{x_2} f(x) \, dx = \int_{u_1}^{u_2} f[\psi(u)] \psi'(u) \, du. \quad (*)$$

It is clear from formula (*) that the integrand expression is transformed just as in the case of an indefinite integral. As regards the limits of integration, the old limits x_1 and x_2 are connected with the new, u_1 and u_2 , in precisely the same way as the old variable x is with the new variable u .

Proof. We transform the indefinite integral $\int f(x) \, dx$ with the aid of the substitution $x = \psi(u)$.

We have in accordance with the rule of Sec. 98:

$$\int f(x) \, dx = \int f[\psi(u)] \psi'(u) \, du = F(u) + C,$$

where $F(u)$ is a primitive of $[f\psi(u)] \psi'(u)$.

* This interval must not be narrower than the interval $[x_1, x_2]$ in which the continuous function $f(x)$ is integrated.

On considering u as a function of x defined by $x = \psi(u)$, we get:

$$\int_{x_1}^{x_2} f(x) dx = F(u) \Big|_{x_1}^{x_2} = F(u|_{x=x_2}) - F(u|_{x=x_1}) = F(u_2) - F(u_1).$$

On the other hand,

$$F(u_2) - F(u_1) = F(u) \Big|_{u_1}^{u_2} = \int_{u_1}^{u_2} f[\psi(u)] \psi'(u) du,$$

and we arrive at equation (*).

Thus, instead of returning, after carrying out the indefinite integration with the aid of a change of variable, to the original variable, then evaluating the integral by double substitution in the given limits, we can make the double substitution directly in the new limits. The result—the value of the definite integral—is the same yet the working involved is less.

After the choice of the necessary substitution $x = \psi(u)$, the only difficulty lies in finding the new limits of integration u_1 and u_2 . These limits are the roots of the equations $x_1 = \psi(u)$ and $x_2 = \psi(u)$ in the unknown u .

Notice that, if $x = \psi(u)$ is not a monotonic function, it may happen that u_1 and u_2 are not defined uniquely (the equations $x_1 = \psi(u)$ and $x_2 = \psi(u)$ have several roots satisfying the conditions of the rule).

Examples. (1) $\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} dx/(x\sqrt{1-x^2})$. We put $x = \sin u$. The new limits of integration u_1 and u_2 are found from the equations $\frac{1}{2} = \sin u$ and $\frac{1}{2}\sqrt{3} = \sin u$; u_1 can be taken equal to $\pi/6$ and u_2 equal to $\pi/3$. As u varies from $\pi/6$ to $\pi/3$, $x = \sin u$ runs right through the interval of integration $[\frac{1}{2}, \frac{1}{2}\sqrt{3}]$. Since, in addition,

$$\frac{dx}{x\sqrt{1-x^2}} = \frac{du}{\sin u},$$

we have

$$\int_{1/2}^{\sqrt{3}/2} \frac{dx}{x\sqrt{1-x^2}} = \int_{\pi/6}^{\pi/3} \frac{du}{\sin u}.$$

The indefinite integral of $1/\sin u$ is equal to $\ln |\tan \frac{1}{2}u| + C$ (Sec. 96, example 9), so that

$$\begin{aligned} \int_{1/2}^{\sqrt{3}/2} \frac{dx}{x\sqrt{1-x^2}} &= \ln \tan \frac{u}{2} \Big|_{\pi/6}^{\pi/3} = \ln \tan \frac{\pi}{6} - \ln \tan \frac{\pi}{12} = \ln \tan \frac{\pi}{6} + \\ &+ \ln \cot \frac{\pi}{12} = \ln \tan \frac{\pi}{6} \cot \frac{\pi}{12} = \ln \frac{\sqrt{3}}{3} (2 + \sqrt{3}) \approx 0.765. \end{aligned}$$

The function $x = \sin u$ is not monotonic, and we can take an interval other than $[\pi/6, \pi/3]$ as the interval of integration. For instance, we also have $x = \sin u = \frac{1}{2}$ with $u = 5\pi/6$, and all the necessary conditions are fulfilled in the interval $[u_1, u_2]$ where $u_1 = 5\pi/6$, $u_2 = 2\pi/3$. But we have in this case:

$$\frac{dx}{x\sqrt{1-x^2}} = -\frac{du}{\sin u}.$$

The minus sign is taken because $\sqrt{1-\sin^2 u} = -\cos u$ for $2\pi/3 \leq u \leq 5\pi/6$. Thus

$$\int_{1/2}^{\sqrt{3}/2} \frac{dx}{x\sqrt{1-x^2}} = - \int_{5\pi/6}^{2\pi/3} \frac{du}{\sin u} = \ln \left| \tan \frac{u}{2} \right| \Big|_{2\pi/3}^{5\pi/6} = \ln \tan \frac{5\pi}{12} - \ln \tan \frac{\pi}{3},$$

and since $\tan 5\pi/12 = \cot \pi/12$, we arrive at the above answer.

(2) Let us find $U_n = \int_0^{\frac{1}{2}\pi} \cos^n x dx$. We put $x = \frac{1}{2}\pi - u$; then

$$\int_0^{\pi/2} \cos^n x dx = - \int_{\pi/2}^0 \cos^n \left(\frac{\pi}{2} - u \right) du = \int_0^{\pi/2} \sin^n u du$$

and since the value of the definite integral does not depend on the notation for the variable of integration, we have

$$\int_0^{\pi/2} \cos^n x dx = \int_0^{\pi/2} \sin^n x dx.$$

Hence all the formulae found in Sec. 106 for $I_n = \int_0^{\frac{1}{2}\pi} \sin^n x dx$ can relate to U_n .

If $n = 2$, then

$$\begin{aligned} \int_0^{\pi/2} \cos^2 x dx &= \int_0^{\pi/2} (1 - \sin^2 x) dx = \frac{\pi}{2} - \int_0^{\pi/2} \sin^2 x dx \\ &= \frac{\pi}{2} - \int_0^{\pi/2} \cos^2 x dx, \end{aligned}$$

whence

$$\int_0^{\pi/2} \cos^2 x \, dx = \int_0^{\pi/2} \sin^2 x \, dx = \pi/4.$$

II. The change of variable in a definite integral is very often performed by using the formula $u = \varphi(x)$, expressing the new variable in terms of the old, instead of by using $x = \psi(u)$. The new limits are given at once in this case by

$$u_1 = \varphi(x_1),$$

$$u_2 = \varphi(x_2).$$

The conditions for validity of the rule for change of variable will certainly be fulfilled here if the function $\varphi(x)$ is monotonic in the interval $[x_1, x_2]$ and has a non-zero derivative. For, the inverse function $x = \psi(u)$ will then be continuous and will have a continuous derivative.

For instance, on putting $\cos x = u$ in the integral

$$I = \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} \, dx,$$

$$\text{we get: } I = - \int_1^0 \frac{du}{1 + u^2} = \arctan u \Big|_0^1 = \pi/4.$$

An example may be mentioned when the substitution rule cannot be used because the conditions for it are not satisfied.

We take

$$\int_{-1}^2 x^2 \, dx \left(= \frac{2^3 - (-1)^3}{3} = 3 \right),$$

and put $x^2 = u$. We have:

$$\int_{-1}^2 x^2 \, dx = \frac{1}{2} \int_1^4 \sqrt{u} \, du = \frac{1}{3} u \sqrt{u} \Big|_1^4 = \frac{7}{3}.$$

The mistake is due to the function $x = \psi(u) = \sqrt{u}$ not satisfying the necessary condition in the interval $[u_1 = 1, u_2 = 4]$: if the positive branch $x = +\sqrt{u}$ is taken, then $x_1 = \psi(u_1) = +\sqrt{1} = +1$, and not -1 , as it should be; whereas if the negative branch $x = -\sqrt{u}$ is taken, then $x_2 = \psi(u_2) = -\sqrt{4} = -2$, and not $+2$, as it should be; thus the rule is inapplicable here.

III. A formula can be obtained, with the aid of the substitution rule, for an integral over a "symmetrical" interval:

$$\int_{-a}^a f(x) dx.$$

We write this integral as:

$$\int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx,$$

and, on changing the variable of integration in the first integral on the right-hand side in accordance with the formula $x = -u$, we get:

$$\int_{-a}^a f(x) dx = \int_0^a f(-u) du + \int_0^a f(x) dx.$$

Hence

$$\int_{-a}^a f(x) dx = \int_0^a [f(-x) + f(x)] dx.$$

The integrand on the right-hand side is zero if $f(x)$ is an odd function, and equal to $2f(x)$ if $f(x)$ is even. Thus

$$\int_{-a}^a f(x) dx = \begin{cases} 0, & \text{if } f(x) \text{ is an odd function,} \\ 2 \int_0^a f(x) dx, & \text{if } f(x) \text{ is an even function.} \end{cases}$$

These formulae are very useful. We shall make some use of them below. We can say at once, for instance, without any evaluations, that

$$\int_{-a}^a x^5 e^{x^2} dx = 0, \quad \int_{-\pi}^{\pi} \sin^3 x \cos^2 x dx = 0.$$

The reader is recommended to give a geometrical interpretation of these formulae.

2. Approximate Methods

108. Numerical integration. We now turn to some useful methods for approximate evaluation of integrals, which enable the approximate value of the integral of any continuous function to be found with sufficient accuracy for practical purposes. The need for an approximate evaluation may arise when no method exists or is

known for finding the accurate value of an integral, or when such a method is known but is laborious and unwieldy. The basis of the approximate methods described here is as follows: if we interpret the integral as the area of a curvilinear trapezoid, an approximate value is obtained for the integral if we find the area of another trapezoid, the boundary curve of which closely follows the curve bounding the original trapezoid. The auxiliary curve is constructed in such a way that it deviates as little as possible from the original curve, whilst at the same time a trapezoid is obtained whose area is easily evaluated.

We shall consider three rules for numerical integration: (1) the rectangle rule, (2) the trapezoid rule, (3) the parabolic trapezoid or Simpson's * rule. The interval of integration is usually divided into equal parts.

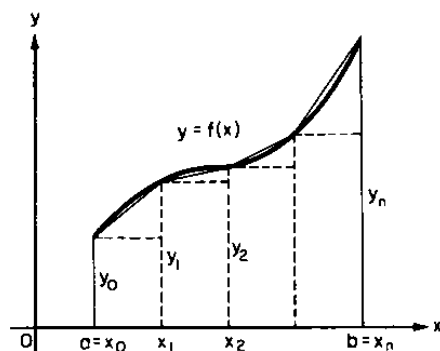


FIG. 119

I. THE RECTANGLE RULE. We divide the interval of integration $[a, b]$ into n equal parts and replace the given trapezoid by the step figure consisting of n rectangles, whose bases are the sub-intervals and whose heights are the ordinates of the curve $y = f(x)$ at the initial (or the final) points of the sub-intervals (Fig. 119). The area of this figure in fact yields the approximate value of the required integral

$$I = \int_a^b y \, dx.$$

If we write $y_0, y_1, \dots, y_{n-1}, y_n$ for the values of $f(x)$ at the points of sub-division, i.e. put $y_k = f(x_k)$, $x_k = a + k \Delta x$, where $\Delta x = \frac{b-a}{n}$,

* Thomas Simpson (1710–1761), an English mathematician.

and k takes the values $0, 1, 2, \dots, n-1$, we obviously have the following formula:

$$I \approx \Delta x (y_0 + y_1 + \dots + y_{n-1}) \quad \text{or} \quad I \approx \Delta x (y_1 + y_2 + \dots + y_n).$$

This is in fact called the "rectangle formula".

As an example, we shall find the value of

$$\int_0^4 x^2 dx \left(= \frac{4^3}{3} \approx 21.33 \right).$$

Let $n = 10$. Then $\Delta x = 0.4$; $x = k \cdot 0.4$ ($k = 0, 1, 2, \dots, 9$) and $y_0 = 0$, $y_1 = (1 \cdot 0.4)^2 = 0.16$, $y_2 = (2 \cdot 0.4)^2 = 0.64$, $\dots$, $y_9 = (9 \cdot 0.4)^2 = 12.96$.

Thus

$$I \approx 0.4(0 + 0.16 + 0.64 + 1.44 + 2.56 + 4 + 5.76 + 7.84 + 10.24 + 12.96) = 0.4 \cdot 45.6 = 18.24.$$

The absolute error is equal to 3.09, and the relative is

$$\frac{3.09 \cdot 100}{21.3} \approx 14.5\%.$$

The error is fairly large, and in order to reduce it, the number of sub-divisions has to be increased substantially, which makes the working laborious.

II. THE TRAPEZOID RULE. We leave the sub-division of the interval as before, but now replace each arc of the curve corresponding to each sub-interval by the chord joining the ends of the arc. The curvilinear trapezoid is thus replaced by n rectilinear trapezoids (Fig. 119). It is geometrically obvious that the area of such a trapezoid more accurately expresses the required area than the area of the n -step figure of the rectangle rule.

The area of each rectilinear trapezoid is equal to the length of the sub-interval $\left(\Delta x = \frac{b-a}{n}\right)$ multiplied by half the sum of the corresponding ordinates. Thus, using the previous notation, we have for the total area:

$$I \approx \Delta x \left(\frac{y_0 + y_1}{2} + \frac{y_1 + y_2}{2} + \dots + \frac{y_{n-1} + y_n}{2} \right),$$

or

$$I \approx \Delta x \left(\frac{y_0 + y_n}{2} + y_1 + y_2 + \dots + y_{n-1} \right).$$

This is in fact called the trapezoid formula. Evaluations using it are no more complicated than with the "rectangle formula", since only one extra value of the function is required, whilst at the same time the accuracy is greater.

Let us apply the "trapezoid formula" to working out of the same integral of x^2 with $n = 10$. We have:

$$I \approx 0.4 \left(\frac{0 + 16}{2} + 0.16 + 0.64 + 1.44 + 2.56 + 4 + 5.76 + 7.84 + \right. \\ \left. + 10.24 + 12.96 \right) = 21.44.$$

The absolute error is now only 0.11, whilst the relative is $\frac{0.11 \cdot 100}{21.33} = 0.51\%$. The accuracy is good, or at any rate sufficient for most practical purposes.

Let us further find the approximate value of

$$I = \int_0^{\pi} \sin x \, dx (= 2).$$

Let $n = 6$. Then

$$I \approx \frac{\pi}{6} \left(\frac{\sin 0 + \sin \pi}{2} + \sin \frac{\pi}{6} + \sin \frac{2\pi}{6} + \sin \frac{3\pi}{6} + \sin \frac{4\pi}{6} + \right. \\ \left. + \sin \frac{5\pi}{6} \right) = \frac{\pi}{6} \left(0.5 + \frac{1}{2} \sqrt{3} + 1 + \frac{1}{2} \sqrt{3} + 0.5 \right) = 1.9541.$$

The absolute error is 0.0459, and the relative 2.5%.

III. SIMPSON'S RULE. This rule in general requires no more labour than the two previous ones, whilst it usually leads to a still more accurate result (with the same sub-division of the interval).

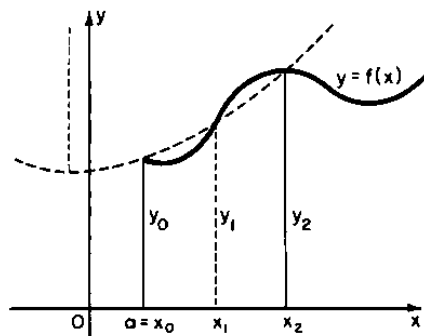


FIG. 120

As before, we divide the interval into n equal parts, but now take n to be even, $n = 2m$. We replace the arc of the curve corresponding to the interval $[x_0, x_2]$ by the arc of a parabola, the axis of which is parallel to the axis of ordinates, and which passes through the following three points: the initial point (x_0, y_0) of the arc, the centre point (x_1, y_1) and the final point (x_2, y_2) of the arc (Fig. 120). A unique parabola of this kind can always be drawn*. We carry out similar replacements in the intervals $[x_2, x_4], [x_4, x_6], \dots, [x_{n-2}, x_n]$.

The given trapezoid is thus replaced by $m = \frac{1}{2}n$ parabolic trapezoids, whose areas are readily found. Let us find the area of the trapezoid bounded by any parabola

$$y = px^2 + qx + r,$$

with axis parallel to the axis of ordinates.

First, let the base of the trapezoid be an interval of Ox symmetrical about the origin: $[-\gamma, \gamma]$. We have for the area of such a parabolic trapezoid:

$$S = \int_{-\gamma}^{\gamma} (px^2 + qx + r) dx = p \int_{-\gamma}^{\gamma} x^2 dx + q \int_{-\gamma}^{\gamma} x dx + r \int_{-\gamma}^{\gamma} dx,$$

i.e.

$$S = p \frac{\gamma^3 - (-\gamma)^3}{3} + q \frac{\gamma^2 - (-\gamma)^2}{2} + r \frac{\gamma - (-\gamma)}{1} = \frac{2}{3} p \gamma^3 + 2r \gamma.$$

We transform this expression thus:

$$\begin{aligned} S &= \frac{2}{3} p \gamma^3 + 2r \gamma = \frac{\gamma}{3} [2p \gamma^2 + 6r] \\ &= \frac{\gamma}{3} [(p \gamma^2 - q \gamma + r) + 4r + (p \gamma^2 + q \gamma + r)]. \end{aligned}$$

We now observe that $p \gamma^2 - q \gamma + r$ is equal to the ordinate y_{in} of the parabola at the initial point $x = -\gamma$ of the trapezoid base, that r is equal to the ordinate y_c of the parabola at the mid-point $x = 0$ of the base, that $p \gamma^2 + q \gamma + r$ is equal to the ordinate y_f of the parabola at the final point $x = \gamma$ of the base; and that, finally, 2γ is equal to the base length. The trapezoid area can thus be written as

$$S = \frac{(\text{base length})}{6} (y_{in} + 4y_c + y_f).$$

* The equation of the parabola has the form $y = px^2 + qx + r$. From the condition that the parabola passes through three points we find three equations which yield the three parameters of the equation: p, q, r .

Obviously, this formula holds for a parabolic trapezoid of the type in question with any base. For, the trapezoid area is unchanged if it is given a parallel displacement such that the base becomes symmetrical about the origin, in which case the area is expressed by our formula.

We return to our original problem and find by this formula the area S_1 of the parabolic trapezoid based on the interval $[x_0, x_2]$:

$$S_1 = \frac{x_2 - x_0}{6} (y_0 + 4y_1 + y_2) = \frac{\Delta x}{3} (y_0 + 4y_1 + y_2),$$

where $\Delta x = \frac{b-a}{n}$. The areas $S_2, S_3, \dots, S_m$ of the following trapezoids are similarly given by

$$S_2 = \frac{\Delta x}{3} (y_2 + 4y_3 + y_4),$$

$$S_3 = \frac{\Delta x}{3} (y_4 + 4y_5 + y_6),$$

$$\dots\dots\dots$$

$$S_m = \frac{\Delta x}{3} (y_{n-2} + 4y_{n-1} + y_n).$$

On adding these equations term by term, we get an expression for the approximate value of the integral:

$$I \approx \frac{\Delta x}{3} [y_0 + y_n + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})].$$

This is in fact Simpson's formula. It can be written alternatively as

$$I \approx \frac{2\Delta x}{3} \left[\left(\frac{y_0 + y_n}{2} + y_1 + y_2 + \dots + y_{n-1} \right) + (y_1 + y_3 + \dots + y_{n-1}) \right].$$

The right-hand side is equal to two thirds the sum of the right-hand sides of the "trapezoid formula" and the "rectangle formula", on condition that all "even" ordinates are neglected in the latter.

For instance, for the integral $I = \int_0^1 x^4 dx (= 0.2)$, we find by Simpson's formula with $n = 10$: $I \approx 0.200013$; the absolute error amounts to 0.000013, and the relative to 0.01%.

Simpson's formula naturally gives an accurate value for the integral of x^3 considered above; and we may mention that it also gives an accurate value for the integral of the cubic function.

Let us apply Simpson's formula to the integral of $\sin x$, found above by the trapezoid formula.

Simple working gives, with $n = 6$:

$$I \approx \frac{2}{3} \left[1.9541 + \frac{\pi}{6} (0.5 + 1 + 0.5) \right] = \frac{2}{3} \left(1.9541 + \frac{\pi}{3} \right) \approx 2.0009.$$

The relative error of the result is 0.4%—a high standard of accuracy, achieved with only six points of sub-division of the interval of integration.

Let us take a further example: $\int_0^1 dx/(1+x^2)$. We have by exact evaluation:

$$\int_0^1 dx/(1+x^2) = \arctan x \Big|_0^1 = \arctan 1 - \arctan 0 = \pi/4.$$

On the other hand, the approximate value of the definite integral may be found by any of the approximation rules; we can thus obtain an approximate value for the number π . Thus, by Simpson's rule, we find even with $n = 4$: $\pi/4 \approx 0.78539$ with all the figures correct.

We shall state without proofs the maximum errors of the above approximation formulae:

$$\left. \begin{array}{l} \text{I. Rectangle, } \frac{\Delta x^2}{24} (b-a) M_2 \\ \text{II. Trapezoid, } \frac{\Delta x^2}{12} (b-a) M_2 \\ \text{III. Parabolic trapezoid, } \frac{\Delta x^4}{180} (b-a) M_4 \end{array} \right\} \text{ where } M_2 \geq |f''(x)| \text{ in } [a, b].$$

III. Parabolic trapezoid, $\frac{\Delta x^4}{180} (b-a) M_4$, where $M_4 \geq |f^{(IV)}(x)|$ in the interval $[a, b]$.

It is clear from this that, when $\Delta x \rightarrow 0$, the maximum error in the first two formulae is an infinitesimal of the second order, whilst it is of the fourth order with the third formula.

Our rules enable us to find approximately the integrals of functions given by graphs or tables as well as by means of formulae.

Example. The width of a river is 20 m; depth measurements every 2 m over a cross-section yield the table:

x	0	2	4	6	8	10	12	14	16	18	20
y	0.2	0.5	0.9	1.1	1.3	1.7	2.1	1.5	1.1	0.6	0.2

Here x denotes the distance in metres from one bank and y the corresponding depth (in metres). We want to find the cross-sectional area of the river.

We find by the "trapezoid formula":

$$S = 2 \left(\frac{0.2 + 0.2}{2} + 0.5 + 0.9 + 1.1 + 1.3 + 1.7 + 2.1 + 1.5 + 1.1 + 0.6 \right) = 22 \text{ m}^2;$$

by Simpson's formula:

$$S = \frac{2}{3} [22 + 2 \cdot (0.5 + 1.1 + 1.7 + 1.5 + 0.6)] = 21.9 \text{ m}^2.$$

The results are very close. We cannot discuss their accuracy, since the problem precludes giving an accurate profile of the river.

109. Graphical integration. We shall now dwell on the graphical method of finding the value of an integral.

Suppose we know the graph of the function $y = f(x)$. We pose the problem: find by geometrical means, without having recourse to any calculations, the integral

$$I = \int_a^b f(x) dx.$$

We shall assume that the scale on Ox is equal to the scale on Oy and that the pole P of the graph is at unit distance (on the scale) from the origin: $OP = 1$ (Fig. 121). We draw the straight line FG , parallel to Ox and cutting the curve $y = f(x)$, such that the area of the rectangle obtained differs as little as possible from the area of our curvilinear trapezoid.

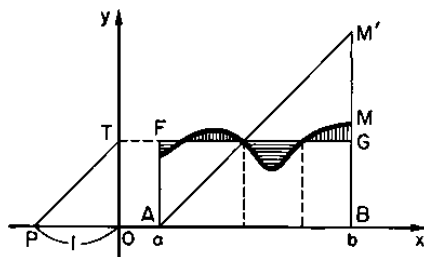


FIG. 121

For this, it is necessary that the area lying between the curve and FG and situated above the straight line should be as far as possible exactly equal to the area between the curve and FG and situated below FG . This is clearly equivalent to establishing approximately the centre line of the trapezoid (see Sec. 91). We continue FG to its intersection with Oy at the point T and join T to the pole. Finally, we draw a straight line parallel to PT from the point A to its intersection with the straight line $x = b$ at M' .

It may easily be shown that the segment $M'B$ represents the required area, i.e. it contains as many linear units of scale as square units are contained in the area of the curvilinear trapezoid. For, we have from the similar triangles POT and ABM' :

$$\frac{BM'}{AB} = \frac{OT}{OP} \quad \text{whence} \quad BM' = \frac{AB \cdot OT}{OP};$$

but $OP = 1$, whilst $AB \cdot OT$ measures the area of the trapezoid.

Unless the interval $[a, b]$ is fairly small, a substantial error will be introduced on drawing FG by eye. To make the construction accurate, we divide the interval of integration into sub-intervals (not necessarily equal) and the complete trapezoid into a series of trapezoids based on these sub-intervals.

We choose the points of sub-division so that the function is monotonic in each sub-interval, and so as to include all the points of intersection of the curve with Ox . If the curve differs insignificantly from a straight line in each sub-interval, the centre-lines of the sub-trapezoids can simply be taken as the ordinates at the mid-points of the sub-intervals*. There is then no need to draw the auxiliary lines by eye. This is what usually happens in practice.

We then construct by the method indicated, for each of the sub-trapezoids in turn, the segment representing its area. For the sake of clarity in the figure, it is handy to mark off this segment on another axis O_1x instead of on Ox , O_1x being parallel to Ox (Fig. 122). At the point $x = a$, the area of the trapezoid measured from the straight line $x = a$ is evidently zero. We mark off on O_1x the point $M'_0(a, 0)$ —it corresponds to the point M_0 of the curve $y = f(x)$. Up to the straight line $x = x_1$, the trapezoid area is equal to the area of the first sub-trapezoid; it is given by the segment M'_1x_1 , which we get if we draw a straight line parallel to PT_1 from the point M'_0 to its intersection with the straight line $x = x_1$ at M'_1 . This point corresponds to the point M_1 on the curve. The area of the trapezoid up to the straight line $x = x_2$ (i.e. the value of the integral from a to x_2) is equal to the sum (in the algebraic sense) of the areas of the first and second sub-trapezoids. It is represented by the segment M'_2x_2 , which is got if we draw from the point M'_1 a straight line parallel to PT_2 to its intersection with the straight line $x = x_2$ at the point M'_2 . This point corresponds to the point M_2 on the curve. On proceeding in this manner, we construct a sequence of points $M'_3, M'_4, \dots$, corresponding to points $M_3, M_4, \dots$ of the curve. The ordinate of the point M'_n corresponding to the point $M'_n(b, f(b))$ of the curve in fact gives us the required value of the integral $I(b)$.

We join the points obtained, $M'_0, M'_1, M'_2, \dots, M'_n$, with a smooth curve. The ordinates of this curve obviously represent approximately the values of the integral from $x = a$ to the corresponding points of the trapezoid base. In other words, it is the graph of the function defined by our integral with variable upper limit:

$$I(x) = \int_a^x f(x) dx.$$

As we know (Sec. 94), the curve $y = I(x)$ is called an integral curve of the function $y = f(x)$. It can be obtained by finding as a preliminary—if this happens to be possible—an analytic expression for the function $y = I(x)$ by means of integration.

Our above geometrical construction of the integral curve from the graph of the integrand is called *graphical integration*.

Just as in the case of graphical differentiation, graphical integration is most convenient when the integrand is given graphically, whilst its analytic ex-

* The greater the number of points of sub-division, the more accurate the construction.

pression is unknown. This quite often happens in practice, as for instance, when the function is given by a graph recorded on some device.

It is clear from the actual process of construction of the integral curve that, in the intervals of variation of x where the curve $y = f(x)$ is located below Ox , i.e. where $f(x)$ is negative, the integral curve is lowered, i.e. the function

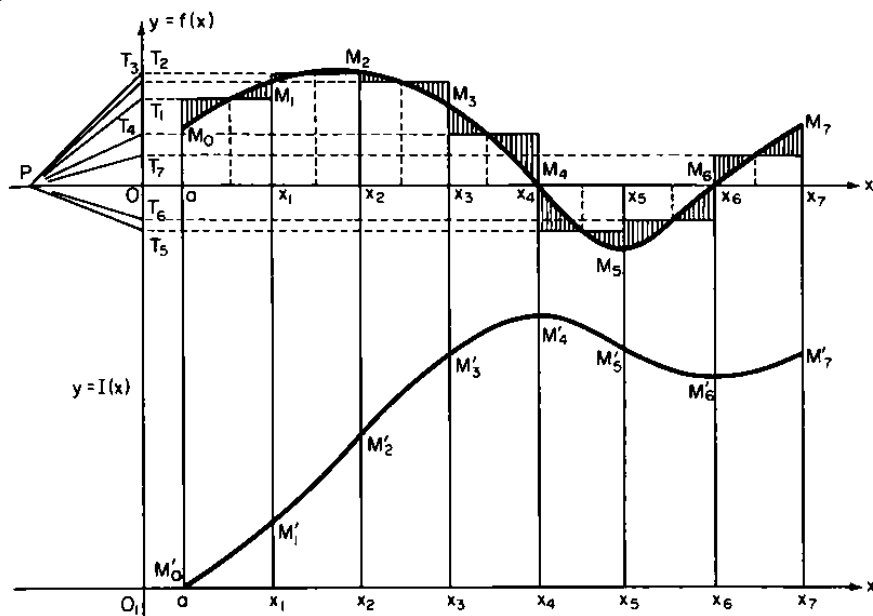


FIG. 122

$y = I(x)$ falls. The reason for this is the fact mentioned above, that the areas corresponding to negative ordinates appear with the minus sign in the value of the integral, i.e. are subtracted. This fact is in full agreement with the fact that $I'(x) = f(x)$.

Our method of graphical integration (drawing the centre lines of the curvilinear trapezoids through the mid-points of their bases) is nothing else but carrying out graphically the method of approximate integration in accordance with the trapezoid rule (Sec. 108, II).

3. Improper Integrals

110. Integrals with infinite limits. The concept of definite integral was established (Chapter V, art. 1) for a finite interval and a function continuous in it. The definition that we gave for this concept becomes meaningless if the interval of integration is infinite or if the function has discontinuities in the interval. As will be seen below, however, it often becomes necessary to extend the concept of integral to cases of an infinite interval of integration and to discontinuous functions.

We turn to the first case.

Let the function $y = f(x)$ be continuous for all values of x , $a \leq x < \infty$. The integral of $f(x)$ has a meaning for any interval $[a, \eta]$, $\eta > a$, and it may naturally be assumed that

$$I(\eta) = \int_a^{\eta} f(x) dx$$

the better expresses the quantity which ought to be taken as the integral of $f(x)$ over the infinite interval $[a, \infty)$, the greater η . We let η increase indefinitely in an arbitrary manner. There are two possibilities: either $I(\eta)$ has a limit as $\eta \rightarrow \infty$ or there is no such limit ($I(\eta)$ tends to infinity or to no limit at all).

Definition. If $\lim_{\eta \rightarrow \infty} I(\eta)$ exists, this limit is called the *improper integral of function $f(x)$ in the interval $[a, \infty)$* and is written as:

$$\int_a^{+\infty} f(x) dx.$$

Thus

$$\int_a^{+\infty} f(x) dx = \lim_{\eta \rightarrow +\infty} \int_a^{\eta} f(x) dx.$$

We say here that the improper integral $\int_a^{+\infty} f(x) dx$ exists or is convergent. If $I(\eta)$ has no limit, we say that the improper integral $\int_a^{+\infty} f(x) dx$ does not exist or is divergent.

Improper integrals may be similarly defined for other infinite intervals:

$$\int_{-\infty}^a f(x) dx = \lim_{\xi \rightarrow -\infty} \int_{\xi}^a f(x) dx,$$

$$\int_{-\infty}^{+\infty} f(x) dx = \lim_{\xi \rightarrow -\infty} \lim_{\eta \rightarrow +\infty} \int_{\xi}^{\eta} f(x) dx = \lim_{\xi \rightarrow -\infty} \int_{\xi}^N f(x) dx + \lim_{\eta \rightarrow \infty} \int_N^{\eta} f(x) dx$$

where N is any number, and ξ and η vary arbitrarily and independently of each other.

The integrals in question are called integrals with infinite limits.

Improper integrals with infinite limits possess all the properties of ordinary (proper) integrals, which are retained in the corresponding passage to the limit.

Let $F(x)$ denote a primitive of $f(x)$. We write conventionally:

$$\int_a^{+\infty} f(x) dx = F(+\infty) - F(a), \quad \int_{-\infty}^a f(x) dx = F(a) - F(-\infty),$$

$$\int_{-\infty}^{+\infty} f(x) dx = F(+\infty) - F(-\infty)$$

where the symbols $F(+\infty)$ and $F(-\infty)$ indicate the limits to which $F(x)$ tends as $x \rightarrow +\infty$ or $x \rightarrow -\infty$.

Suppose that the curve $y = f(x)$ is the boundary of an "infinite trapezoid" (a "trapezoid" with infinite base) (Fig. 123). If the improper integral of $f(x)$ exists, evaluated over the trapezoid base, it may naturally be supposed that it measures the "area" of this infinite trapezoid; otherwise it would not be possible to speak of the area of the trapezoid.

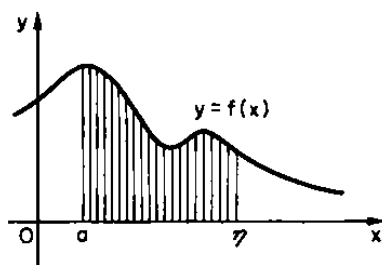


FIG. 123

The existence of the improper integral can be explained visually by the fact that indefinite increase in the base is bound up with indefinite and fairly rapid decrease in the height of the trapezoid, whilst if the integral does not exist, increase of the base is not "compensated" by a rapid enough decrease in the height.

For instance, the infinite trapezoid bounded by positive Ox , the straight line $x = a (> 0)$ and the curve $y = 1/x^3$ can be ascribed an area equal to $1/2a^2$, because

$$\int_a^{+\infty} \frac{1}{x^3} dx = -\frac{1}{2} \frac{1}{x^2} \Big|_a^{+\infty} = \frac{1}{2a^2}.$$

The infinite trapezoid bounded say by the hyperbola $y = 1/x$, positive Ox and the straight line $x = a (> 0)$ cannot be ascribed an

area, since

$$\int_a^{+\infty} \frac{1}{x} dx = \ln x \Big|_a^{+\infty} = \infty.$$

In general, it may easily be seen that

The trapezoid with base $[a, +\infty]$, $a > 0$, bounded by the curve $y = 1/x^m$ ($m > 0$), has a "finite" area or has no area (i.e. has an "infinite area") depending on whether $m > 1$ or $m \leq 1$.

We shall use this for deducing convergence and divergence tests for improper integrals of the type in question.

Examples. (1) Let us evaluate the area of the curvilinear trapezoid bounded by the whole of Ox and the curve $y = 1/(1+x^2)$.

We have:

$$S = \int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \arctan x \Big|_{-\infty}^{+\infty} = \pi.$$

However, this improper integral can be transformed to an ordinary one by the substitution $x = \tan u$:

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\pi/2}^{+\pi/2} \frac{du}{\cos^2 u \frac{1}{\cos^2 u}} = \int_{-\pi/2}^{+\pi/2} du = \pi.$$

(2) If a particle M of mass m , situated at the origin, attracts a free particle M_1 of mass 1 at a distance x from M on Ox , the attractive force P is given by the familiar formula

$$P = k \frac{m}{x^2}, \quad \text{where } k \text{ is constant,}$$

whilst the work done on displacing M_1 from the point $x = r$ to $x = a$ ($a > r > 0$) is given by

$$A = - \int_r^a k \frac{m}{x^2} dx = k m \left(\frac{1}{a} - \frac{1}{r} \right);$$

the "minus" is put in front of the integral because the direction of the force is opposite to the direction of motion of the particle M (the work is negative for the same reason).

If $a = \infty$, then

$$A = - \int_r^{\infty} k \frac{m}{x^2} dx = - k \frac{m}{r}.$$

If the particle M_1 is displaced "from infinity" to the point $x = r$, the force of Newtonian attraction performs the positive work

$$A = k \frac{m}{r},$$

expressing the degree to which potential energy is accumulated at the point $x = r$. It is called the potential of the force of attraction of the material particle M at $x = r$ (or at the point $x = r$).

We must mention two important improper integrals whose values will be found by us later (see part II, Secs. 180 and 181):

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi} \quad (\text{Poisson's integral}),$$

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} \quad (\text{Dirichlet's integral}).$$

It is of interest that the corresponding indefinite integrals are not expressible in elementary functions.

III. Tests for convergence and divergence of integrals with infinite limits. We are usually concerned with improper integrals of functions whose primitives are unknown; at the same time, it may be important for us to know whether or not the integral exists. We have recourse to tests which enable us to discover whether the integral is convergent or divergent without finding the primitive.

Let the integrand $f(x)$ be non-negative in the interval of integration: $f(x) \geq 0$. The following sufficient tests now hold for the convergence and divergence of the integral with infinite limits.

CONVERGENCE TEST. If constants $M > 0$ and $m > 1$ exist such that, for $0 < a \leq x < \infty$,

$$f(x) \leq \frac{M}{x^m}$$

the integral $\int_a^\infty f(x) dx$ is convergent.

Proof. We have, with $\eta > a$,

$$I(\eta) = \int_a^\eta f(x) dx \leq \int_a^\eta \frac{M}{x^m} dx = \frac{M}{m-1} \left(\frac{1}{a^{m-1}} - \frac{1}{\eta^{m-1}} \right) <$$

$$< \frac{M}{(m-1)a^{m-1}};$$

the function $I(\eta)$ is therefore bounded, and since it is increasing with increasing η (because $f(x)$ is non-negative), it has a limit as $\eta \rightarrow \infty$.

DIVERGENCE TEST. If constants $M > 0$ and $m \leq 1$ exist, such that, with $0 < a \leq x < \infty$,

$$f(x) \geq \frac{M}{x^m}$$

the integral $\int_a^\infty f(x) dx$ is divergent.

Proof. We have, with $\eta > a$,

$$I(\eta) = \int_a^\eta f(x) dx \geq \int_a^\eta \frac{M}{x^m} dx = \begin{cases} \frac{M}{1-m} (\eta^{1-m} - a^{1-m}) & \text{for } m < 1, \\ M \ln \frac{\eta}{a} & \text{for } m = 1. \end{cases}$$

Both the expressions obtained tend to infinity as $\eta \rightarrow \infty$.

We see that the convergence of the integral is guaranteed when the integrand tends fast enough to zero on indefinite increase of its argument, and that, conversely, if it tends insufficiently rapidly to zero, the integral must be divergent.

For instance, we conclude from the convergence test that

$$\int_1^\infty \frac{dx}{\sqrt{1+x^2} \sqrt[3]{1+x^3}}$$

is convergent, because

$$\frac{1}{\sqrt{1+x^2} \sqrt[3]{1+x^3}} < \frac{1}{x^2}$$

throughout the interval $(1, \infty)$.

We conclude by the divergence test that

$$\int_2^\infty \frac{\sqrt[3]{x^2+1}}{\sqrt{x^2-1}} dx$$

does not exist, because

$$\frac{\sqrt[3]{x^2+1}}{\sqrt{x^2-1}} > \frac{x^{2/3}}{x} = \frac{1}{x^{1/3}}.$$

These tests lead us to the following further proposition (with $f(x) \geq 0$).

SUFFICIENT TESTS. If $\lim_{x \rightarrow +\infty} x^m f(x) = A$, then, with $m > 1$, the integral

$$\int_a^\infty f(x) dx$$

is convergent, whilst it is divergent with $m \leq 1$ and $A > 0$.

Proof. Given an arbitrarily small positive σ , we have by the familiar properties of limits:

$$x^m f(x) < A + \sigma$$

for all $x > N$, where N is a sufficiently large positive number corresponding to the chosen σ .

Hence

$$f(x) < \frac{M}{x^m}, \quad M = A + \sigma.$$

We write:

$$\int_a^\infty f(x) dx = \int_a^N f(x) dx + \int_N^\infty f(x) dx. \quad (\dagger)$$

The first integral on the right-hand side of this equation exists as an ordinary ("proper") integral, whilst the second exists with $m > 1$ on account of satisfying the conditions of the fundamental test.

If $A > 0$, we also have, for $x > N$, $A - \sigma < x^m f(x)$, where $0 < \sigma < A$, or

$$f(x) > \frac{M}{x^m}, \quad M = A - \sigma, \quad 0 < \sigma < A$$

and the second integral on the right-hand side of equation ($\dagger$) does not exist with $m \leq 1$ by virtue of the basic divergence test (it is equal to $+\infty$), i.e. the original integral is divergent (it is equal to $+\infty$ also).

We shall give one extremely general test for the cases when the integrand changes sign in the infinite interval of integration.

SUFFICIENT TEST FOR CONVERGENCE. If the integral $\int_a^\infty |f(x)| dx$ is convergent, $\int_a^\infty f(x) dx$ is also convergent, and is said in this case to be absolutely convergent.

Proof. We bring in two auxiliary positive functions $f^+(x)$ and $f^-(x)$, defined as follows:

$$f^+(x) = \frac{|f(x)| + f(x)}{2},$$

$$f^-(x) = \frac{|f(x)| - f(x)}{2}.$$

We see that, given x , $f^+(x)$ is equal to $f(x)$ if $f(x) \geq 0$, and is equal to 0 if $f(x) < 0$, whilst $f^-(x)$ is equal to $|f(x)|$ if $f(x) \leq 0$ and equal to 0 if $f(x) > 0$. (The reader is recommended to draw the graphs of $f^+(x)$ and $f^-(x)$, given the graph of $f(x)$).

We have:

$$\begin{aligned}|f(x)| &= f^+(x) + f^-(x), \\ f(x) &= f^+(x) - f^-(x)\end{aligned}$$

so that

$$\begin{aligned}\int_a^\eta |f(x)| dx &= \int_a^\eta f^+(x) dx + \int_a^\eta f^-(x) dx, \\ \int_a^\eta f(x) dx &= \int_a^\eta f^+(x) dx - \int_a^\eta f^-(x) dx.\end{aligned}$$

By hypothesis, the integral of $|f(x)|$ has a limit as $\eta \rightarrow \infty$; it is easily shown that the integrals on the right-hand side of the first equation consequently also have limits. For, each of them is a positive increasing function of η ; if one of them had no limit, which could only happen if it tended to $+\infty$, the sum of the integrals would clearly also have no limit and would also tend to $+\infty$, which contradicts our hypothesis. But if the integrals of $f^+(x)$ and $f^-(x)$ have limits as $\eta \rightarrow \infty$, their difference, i.e. by the second equation, the given integral of $f(x)$, also has a limit as $\eta \rightarrow \infty$, in other words, it is convergent.

If the integral of $|f(x)|$ is divergent, on the basis of this alone nothing further can be said regarding the integral of $f(x)$: it may be divergent or convergent; in the latter case it is said to be non-absolutely or conditionally convergent.

Example. $\int_2^\infty \frac{\sin x \ln x}{x \sqrt{x^2 - 1}} dx$ is absolutely convergent. For, if $2 \leq x < \infty$, then

$$\left| \frac{\sin x \ln x}{x \sqrt{x^2 - 1}} \right| \leq \frac{1 \cdot x^\varepsilon}{x^2 \sqrt{1 - \frac{1}{x^2}}} \leq \frac{\frac{2}{\sqrt{3}}}{x^{2-\varepsilon}}, \quad \text{where } 2 - \varepsilon > 1.$$

112. Integral of a function with infinite jumps. If a function $f(x)$ has a number of points of finite discontinuity in the interval $[a, b]$, no difficulties are involved in defining the integral of the function. In fact, it is natural to assume here that *the integral is simply the sum of the ordinary (proper) integrals over the sub-intervals into which $[a, b]$ is divided by all the points of discontinuity of the function.* We write $c_1, c_2, \dots, c_k$ for these, $a < c_1 < c_2 < \dots < c_k < b$; then

$$\int_a^b f(x) dx = \int_a^{c_1} f(x) dx + \int_{c_1}^{c_2} f(x) dx + \dots + \int_{c_k}^b f(x) dx.$$

This establishes a completely natural definition of the "area" of the curvilinear trapezoid corresponding to $y = f(x)$ with a finite number of points of finite discontinuity in $[a, b]$ (Fig. 124): *the area of such a trapezoid is the sum of the areas of the trapezoids based on the sub-intervals $[a, c_1], [c_1, c_2], \dots, [c_k, b]$ lying between the successive points of discontinuity.*

We turn to an extension of the concept of integral to a function with infinite discontinuities.

Let $y = f(x)$ be continuous for all x , $a \leq x < b$, and have an infinite discontinuity at the end $x = b$ of the interval. It is clear that the ordinary definition of integral becomes meaningless here. But if we take the ordinary integral

$$I(\varepsilon) = \int_a^{b-\varepsilon} f(x) dx, \quad \varepsilon > 0,$$

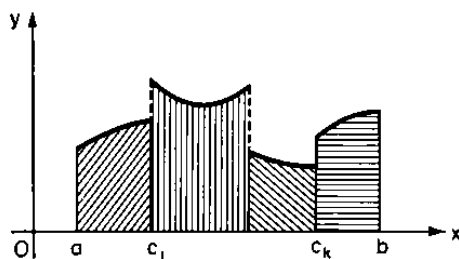


FIG. 124

we ought to agree, as in the previous case (Sec. 110), that $I(\varepsilon)$ better and better approximates with decreasing ε to the quantity that should be taken as the integral of $f(x)$ in the interval $[a, b]$. We let ε tend to zero in an arbitrary manner. Then $I(\varepsilon)$ either has a limit or no such limit exists (the integral tends to infinity or has no sort of limit).

Definition. If $\lim_{\varepsilon \rightarrow 0} I(\varepsilon)$ exists, this limit is called the **improper integral** of $f(x)$ in the interval $[a, b]$ and is written as $\int_a^b f(x) dx$.

Thus

$$\int_a^b f(x) dx = \lim_{\varepsilon \rightarrow 0} \int_a^{b-\varepsilon} f(x) dx.$$

We say here that the improper integral $\int_a^b f(x) dx$ exists or is convergent. Whereas if $I(\varepsilon)$ has no limit, the improper integral $\int_a^b f(x) dx$ is said to be non-existent or divergent.

Similarly, if $f(x)$ has an infinite discontinuity only at the left-hand end $x = a$ of the interval, we have

$$\int_a^b f(x) dx = \lim_{\delta \rightarrow 0} \int_{a+\delta}^b f(x) dx, \quad \delta > 0.$$

Finally, if $f(x)$ has an infinite discontinuity at some intermediate point $x = c$ of $[a, b]$, $a < c < b$, then

$$\int_a^b f(x) dx = \lim_{\varepsilon \rightarrow 0} \int_a^{c-\varepsilon} f(x) dx + \lim_{\delta \rightarrow 0} \int_{c+\delta}^b f(x) dx,$$

where ε and δ tend to zero arbitrarily and independently of each other.

It is completely obvious how the improper integral is to be defined in the case of a finite number of points of infinite discontinuity of the integrand in the interval of integration.

All the properties of ordinary integrals may be carried over to improper integrals of discontinuous functions, these properties being retained during the relevant passage to the limit.

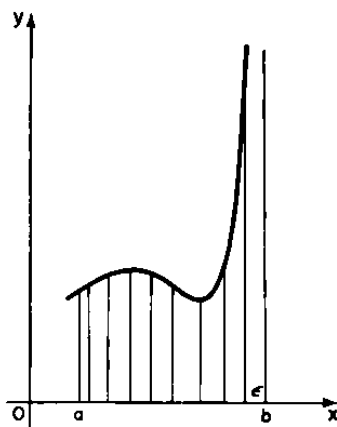


FIG. 125

Suppose that the curve $y = f(x)$ has a number of asymptotes perpendicular to Ox ; then the trapezoid bounded by it will be infinite (will have "infinite heights") (Fig. 125). If the improper integral of $f(x)$ exists, it is regarded as measuring the "area" of this infinite trapezoid; otherwise, the trapezoid has no area.

The existence of the improper integral can also be explained visually here by the fact that, in the one case, indefinite increase of the height is "compensated" by a fast enough decrease in the width of the "needle", whilst in the other case it is not "compensated".

For instance, the infinite trapezoid bounded by the curve $y = 1/\sqrt{x}$ and the straight lines $x = 0$, $x = a$, $y = 0$, can be assigned an area

equal to $2\sqrt{a}$, because

$$\int_0^a \frac{1}{\sqrt{x}} dx = \lim_{\delta \rightarrow 0} \int_{\delta}^a \frac{1}{\sqrt{x}} dx = \lim_{\delta \rightarrow 0} 2(\sqrt{a} - \sqrt{\delta}) = 2\sqrt{a}.$$

The infinite trapezoid bounded say by the hyperbola $y = 1/x$ and the same straight lines $x = 0$, $x = a$, $y = 0$, cannot be assigned an area, because

$$\int_0^a \frac{1}{x} dx = \lim_{\delta \rightarrow 0} \int_{\delta}^a \frac{1}{x} dx = \lim_{\delta \rightarrow 0} \ln \frac{a}{\delta} = \infty.$$

It may readily be seen that, in general:

The trapezoid with base $[a, b]$, bounded by the curve $y = 1/(x - a)^m$ or $y = 1/(b - x)^m$ ($m > 0$), has a ("finite") area or has no area (i.e. has an "infinite area") depending on whether $m < 1$ or $m \geq 1$.

We shall later make use of this for deducing a test for the convergence and divergence of improper integrals of this last type.

Example. Let us find the area S of the "infinite needle" bounded by Ox , the straight lines $x = a$ ($a < 0$), $x = b$ ($b < 0$) and the curve $y = 1/\sqrt[3]{x^2}$. Since the function has an infinite discontinuity in the interval (at $x = 0$), we have

$$\begin{aligned} S &= \int_a^b \frac{dx}{\sqrt[3]{x^2}} = \lim_{\epsilon \rightarrow 0} \int_a^{-\epsilon} \frac{dx}{\sqrt[3]{x^2}} + \lim_{\delta \rightarrow 0} \int_{\delta}^b \frac{dx}{\sqrt[3]{x^2}} \\ &= -3\sqrt[3]{a} + 3\sqrt[3]{b} = 3(\sqrt[3]{b} - \sqrt[3]{a}). \end{aligned}$$

The last formula may suggest the erroneous conclusion that it is unnecessary to split the integral into two before passing to the limit etc., since the same result is obtained directly:

$$\int_a^b \frac{dx}{\sqrt[3]{x^2}} = 3\sqrt[3]{x} \Big|_a^b = 3(\sqrt[3]{b} - \sqrt[3]{a}).$$

This conclusion can be seen to be completely false by taking

$$\int_a^b \frac{dx}{x^2}, \quad a < 0 < b;$$

we find on evaluating the integral by the ordinary rule:

$$\int_a^b \frac{dx}{x^2} = -\frac{1}{x} \Big|_a^b = \frac{1}{a} - \frac{1}{b},$$

i.e. the integral of a positive function is a negative number!

This clearly absurd result is due to the interval of integration $[a, b]$ containing a point $x = 0$ of infinite discontinuity of the integrand such that the integrals $\int_a^0 dx/x^2$ and $\int_0^b dx/x^2$ do not exist.

Consequently, in cases when the function has infinite discontinuities inside or at the ends of the interval of integration, one must proceed warily and pay strict heed to the extension described above of the concept of integral to these cases.

113. Tests for convergence and divergence of the integrals of discontinuous functions. The following sufficient tests hold for the convergence and divergence of an improper integral of a function with an infinite discontinuity at $x = b$ in the case when the integrand is non-negative in the interval of integration $[a, b]$.

CONVERGENCE TEST. If constants $M > 0$ and $0 < m < 1$ exist such that, for $0 < a \leq x < b$,

$$f(x) \leq \frac{M}{(b-x)^m},$$

the integral $\int_a^b f(x) dx$ is convergent.

Proof. We have, with $0 < \varepsilon < b - a$:

$$\begin{aligned} I(\varepsilon) &= \int_a^{b-\varepsilon} f(x) dx \leq \int_a^{b-\varepsilon} \frac{M}{(b-x)^m} dx \\ &= \frac{M}{1-m} [(b-a)^{1-m} - \varepsilon^{1-m}] < \frac{M(b-a)^{1-m}}{1-m}; \end{aligned}$$

consequently the function $I(\varepsilon)$ is bounded, and since it is increasing with decreasing ε (by virtue of $f(x)$ being non-negative), it has a limit as $\varepsilon \rightarrow 0$.

DIVERGENCE TEST. If constants $M > 0$ and $m \geq 1$ exist such that, with $0 < a \leq x < b$,

$$f(x) \geq \frac{M}{(b-x)^m},$$

the integral $\int_a^b f(x) dx$ is divergent.

Proof. We have, with $0 < \varepsilon < b - a$:

$$\begin{aligned} I(\varepsilon) &= \int_a^{b-\varepsilon} f(x) dx \geq \int_a^{b-\varepsilon} \frac{M}{(b-x)^m} dx \\ &= \begin{cases} \frac{M}{m-1} \left(\frac{1}{\varepsilon^{m-1}} - \frac{1}{(b-a)^{m-1}} \right) & \text{with } m > 1, \\ M \ln \frac{b-a}{\varepsilon} & \text{with } m = 1. \end{cases} \end{aligned}$$

Both these expressions tend to infinity as $\varepsilon \rightarrow 0$.

We see that the convergence of the integral is guaranteed by not too rapid an increase of the integrand when its argument tends to the point of infinite discontinuity, and that a fast enough increase, on the contrary, ensures divergence of the integral.

For instance, it can be seen by the convergence test that

$$\int_0^1 \frac{dx}{\sqrt{1-x^2} \sqrt[3]{1-x^3}}$$

is convergent, because

$$\frac{1}{\sqrt{1-x^2} \sqrt[3]{1-x^3}} < \frac{1}{(1-x)^{5/6}}$$

throughout the interval $[0, 1]$.

We see by the divergence test that

$$\int_0^1 \frac{dx}{\sin^2(1-x)}$$

does not exist, because

$$\frac{1}{\sin^2(1-x)} > \frac{1}{(1-x)^2}$$

throughout the interval $[0, 1]$.

The tests can obviously be carried over automatically to the case when the integrands have discontinuities at the lower limit.

A further proposition follows from the above tests (with $f(x) > 0$).

SUFFICIENT TESTS. If $\lim_{x \rightarrow b} (b-x)^m f(x) = A$,

then, with $0 < m < 1$, the integral $\int_a^b f(x) dx$

is convergent, whilst with $m \geq 1$ and $A > 0$, it is divergent.

Proof. We have by the familiar properties of limits, for any given arbitrarily small positive σ :

$$(b-x)^m f(x) < A + \sigma$$

for all x satisfying the condition $0 < b-x < \nu$, where ν is a sufficiently small number corresponding to the chosen σ . Hence

$$f(x) < \frac{M}{(b-x)^m}, \quad M = A + \sigma.$$

We write:

$$\int_a^b f(x) dx = \int_a^{b-\nu} f(x) dx + \int_{b-\nu}^b f(x) dx. \quad (\dagger)$$

The first integral on the right-hand side of this equation exists as an ordinary ("proper") integral, whilst the second, with $0 < m < 1$, exists due to satisfying the conditions of the basic convergence test.

Whereas if $A > 0$, we have in the interval $[b-\nu, b]$:

$$f(x) > \frac{M}{(b-x)^m}, \quad M = A - \sigma, \quad 0 < \sigma < A,$$

and the second integral on the right-hand side of ($\dagger$) does not exist (is equal to $+\infty$) with $m \geq 1$ by virtue of the basic divergence test, i.e. the original integral is also divergent (it is also equal to $+\infty$).

On now returning to cases when the integrand changes sign in the interval of integration, a general test can be given analogous to the above.

SUFFICIENT TEST FOR CONVERGENCE. If $\int_a^b |f(x)| dx$ is convergent, $\int_a^b f(x) dx$ is also convergent, and is said to be absolutely convergent.

Proof. Let $f(x)$ be continuous in $[a, b)$, and let $\lim_{x \rightarrow b} f(x) = \infty$. We bring in functions $f^+(x)$ and $f^-(x)$ (see Sec. 111), and write

$$\begin{aligned} \int_a^{b-\varepsilon} |f(x)| dx &= \int_a^{b-\varepsilon} f^+(x) dx + \int_a^{b-\varepsilon} f^-(x) dx, \\ \int_a^{b-\varepsilon} f(x) dx &= \int_a^{b-\varepsilon} f^+(x) dx - \int_a^{b-\varepsilon} f^-(x) dx. \end{aligned}$$

On arguing further precisely as in Sec. 111, we conclude that, since the integral on the left-hand side of the first equation has a limit as $\varepsilon \rightarrow 0$, each of the integrals on the right-hand side must also have a limit, i.e. the integral on the left-hand side of the second equation has a limit.

When the integral of the absolute value of a function is divergent, the integral of the function itself may be either convergent or divergent.

APPLICATIONS OF THE INTEGRAL

1. Elementary Problems and Methods of Solution

114. Method of "summation of elements". We considered in Chapter V (Secs. 84, 85) the problems of finding (1) the area of a curvilinear trapezoid, (2) the work done by a variable force, (3) the distance travelled by a moving body, (4) the mass of a material curve. By generalizing the mathematical operations with the aid of which these problems were solved, we arrived at the concept of integral.

In order to characterize in the most general way the simplest class of problems leading to an integral, and to form a scheme for their solution, we shall carefully scrutinize once more the arguments which enabled us to express the solutions of these four different types of physical problem in the same mathematical form, as integrals.

First of all, in all four problems the required magnitude is related to an interval $[a, b]$ of variation of the independent variable x , $a \leq x \leq b$. For instance, we seek the area of the curvilinear trapezoid having $[a, b]$ as its base; we seek the work done over the piece of path $[a, b]$; we seek the distance traversed by the body in the time interval $[a, b]$; and we seek the mass disposed on the segment $[a, b]$ of the curve.

Our first step is to split up the interval into sub-intervals $[x_i, x_{i+1}]$, $x_i \leq x \leq x_{i+1}$ ($i = 0, 1, \dots, n - 1$). We start here from the essential premise that the magnitude to be found, corresponding to the whole of the interval, can be formed as the sum of the similar magnitudes corresponding to all the sub-intervals. This property is called *additiveness*. Area, work, distance, mass are additive with respect to the intervals in which they are defined: the area of the curvilinear trapezoid having $[a, b]$ as its base is the sum of the areas of all the curvilinear trapezoids having as bases the intervals $[x_i, x_{i+1}]$; the work done by a force over the piece of path $[a, b]$ is equal to the

sum of all the works done by the force in the pieces $[x_i, x_{i+1}]$, and so on. Naturally, the integral to which these magnitudes lead has the property of additiveness with respect to the interval of integration. This is in fact expressed in property IV of the integral (Sec. 89).

We shall consider the required magnitude (area, work, path, mass) in a variable interval $[\alpha, x]$ with a fixed left-hand end α . This magnitude will now be some function $F(x)$ of the abscissa x of the right-hand end. In accordance with this, our magnitude, referred to the interval $[a, b]$, $\alpha \leq a < b$, will be equal to $F(b) - F(a)$, whilst the magnitude, referred to the interval $[x_i, x_{i+1}]$, will be equal to $\Delta F(x_i) = F(x_{i+1}) - F(x_i)$. For, the interval up to x_{i+1} is the sum of the interval $[\alpha, x_i]$ and the interval $[x_i, x_{i+1}]$, so that, by virtue of its additiveness, $F(x_{i+1})$ is the sum of $F(x_i)$ and the magnitude corresponding to $[x_i, x_{i+1}]$, i.e. the latter is equal to the difference $F(x_{i+1}) - F(x_i)$.

Hence, if we split the given interval into parts, we start out from the fact that

$$F(b) - F(a) = \sum_{i=0}^{n-1} [F(x_{i+1}) - F(x_i)] = \sum_{i=0}^{n-1} \Delta F(x_i). \quad (\dagger)$$

The part $\Delta F(x_i)$ of our magnitude is called the true element of the magnitude corresponding to the sub-interval $[x_i, x_{i+1}]$. Equation($\dagger$) in itself expresses the trivial fact that the whole is equal to the sum of all its parts, and yields nothing for the solution of the problem: the increments $\Delta F(x_i)$ of $F(x)$, as well as $F(x)$ itself, are not previously known to us.

But by writing the required magnitude as the sum of its true elements we prepare the way for the next decisive step.

We notice that $F(x)$ is always defined by some known magnitude $f(x)$ (the ordinate of a curve, bounding the trapezoid; the force acting; the velocity of the motion; the density), such that, if $f(x)$ remains constant in an interval, $F(x)$ is equal to the product of the constant $f(x)$ and the length of the interval. For instance, with constant ordinate the trapezoid is a rectangle and its area is equal to the product of the ordinate and the interval length; for a constant force, the work is equal to the force times the path length, and so on.

The decisive step in the argument consists in replacing the true element of the required magnitude corresponding to a sub-interval by the magnitude obtained on the assumption that the definitive magnitude $f(x)$ remains constant in the sub-interval, the constant being its value at say the left-hand end of $[x_i, x_{i+1}]$ i.e. $f(x) = f(x_i)$.

In other words, $\Delta F(x_i)$ is replaced by $f(x_i) \Delta x_i$ ($\Delta x_i = x_{i+1} - x_i$), and instead of the accurate equation (†) we get the merely approximate equation

$$F(b) - F(a) \approx \sum_{i=0}^{n-1} f(x_i) \Delta x_i.$$

But we suppose that, on indefinite increase in the number of sub-intervals and on their lengths tending to zero, the error in this approximation tends to zero; on passing to the limit, i.e. to the integral, we arrive at the equation

$$F(b) - F(a) = \int_a^b f(x) dx,$$

giving us the expression for the required magnitude.

The significance of the striking replacement of the increment $\Delta F(x_i)$ by the expression $f(x_i) \Delta x_i$ lies in the fact that, whereas the former is unknown to us, the latter is known, since the actual statement of the problem tells us the equation of the curve, or the expression for the force as a function of the path, etc.

The magnitude $f(x_i) \Delta x_i$, replacing the true element $\Delta F(x_i)$ of the required magnitude, is called the element corresponding to the sub-interval $[x_i, x_i + \Delta x_i]$.

We see that the method which we used for solving the above four problems consists in replacing summation of the true elements of the required magnitude $\Delta F(x)$ by integration of its "element" $f(x) dx$, or, as it is described conventionally, by "summing its elements"*.

Of course the method of "summation of elements" can be used directly in other problems, but there is nothing to be gained by this, since it leads essentially to a repetition in each case of the definition of integral, which has been established once and for all time; moreover it lacks reliability, since it contains no criterion for the correctness of the choice of the "element" replacing the "true element" of the magnitude.

The theory of the definite integral enables us to give a simpler and more reliable method.

115. Method of "differential equation". Scheme for solution of problems. Suppose u is a given magnitude:

* It must be remembered, when using the figurative term "summation", that we are speaking about the operation of integration, and not of summation in the straightforward sense.

(1) dependent on an interval (thus the area of the curvilinear trapezoid depends on the interval which forms its base; the work depends on the piece of path over which it is expended, etc.). If we relate u to a variable interval say with fixed left-hand end α and variable right-hand end x , u can be regarded as a function of the independent variable x , i.e. $u = F(x)$;

(2) having the property of additiveness, i.e. when the interval of variation of x is sub-divided, the magnitude u corresponding to the whole of the interval is formed from the sum of the parts corresponding to the sub-intervals;

(3) differentiable as a function of x in the interval $[a, b]$ in question.

Finally, suppose we want to find the u corresponding to the interval $[a, b]$ of variation of the independent variable.

The above three requirements in fact characterize the problems whose solutions can be expressed with the aid of the integral.

The required value is equal to $F(b) - F(a)$, whilst we have by the Newton-Leibniz formula:

$$F(b) - F(a) = \int_a^b dF(x).$$

Therefore,

The required magnitude is measured by the integral of the differential of the unknown function $F(x)$ over the relevant interval.

The same thing is readily obtained from (†) of Sec. 114. We have by Lagrange's formula:

$$\Delta F(x_i) = F'(\xi_i) \Delta x_i, \quad x_i \leq \xi_i \leq x_{i+1},$$

i.e.

$$F(b) - F(a) = \sum_{i=0}^{n-1} F'(\xi_i) \Delta x_i.$$

On passing to the limit, with the assumption that $\max |\Delta x_i| \rightarrow 0$, we obtain by the existence theorem for the integral (Sec. 86):

$$F(b) - F(a) = \lim_{\max |\Delta x_i| \rightarrow 0} \sum_{i=0}^{n-1} F'(\xi_i) \Delta x_i = \int_a^b F'(x) dx = \int_a^b dF(x). *$$

Although the function $u = F(x)$ is unknown, its differential is often expressible in terms of functions given by the problem data and the differential of the independent variable; thus the entire

* It must be pointed out that we have here deduced the Newton-Leibnitz formula (Sec. 93) by a different method to the previous one.

problem reduces to evaluating the integral of some known function. This was the case in all four of the problems discussed in Sec. 114.

Definition. The equation

$$du = f(x) dx$$

connecting the differential of the unknown function with the differential of the independent variable, is called the differential equation of the problem.

Thus, when solving any concrete problem of the type laid down, we can avoid the whole of the discussion justifying the expression of the solution as an integral, and use its final deduction that *the required magnitude is equal to the integral of the differential of the unknown function $F(x)$.*

The solution of problems splits up naturally into three steps:

1. Formation of the differential equation, i.e. seeking the expression for the differential of the required magnitude u :

$$du = f(x) dx \quad (*)$$

where $f(x)$ is a known function.

For this, an infinitesimal increment dx is taken at an arbitrary point x of the given interval $[a, b]$; the differential $dF(x)$ is found by using the facts that (1) it is the principal part of the increment ΔF (i.e. of the true element of the magnitude corresponding to the interval $[x, x + dx]$) and (2) it is proportional to dx . In concrete problems the differential $dF(x)$ is often found as the increment of $F(x)$ evaluated on the assumption that the magnitudes defining it preserve their values at x throughout $[x, x + dx]$.

For instance, the differential at x of the area of the curvilinear trapezoid is the area of the rectangle with base $[x, x + dx]$ and height equal to the value at x of the ordinate of the curve bounding the trapezoid.

It can be verified each time that the expression obtained is in fact the differential by showing that it has the two familiar properties of a differential (Sec. 51).

II. Passage from the differential equation(*) to the integral

$$F(b) - F(a) = \int_a^b f(x) dx. \quad (**)$$

This passage is accomplished by "summation of all the elements", which leads to the accurate result.

Limits a and b of integral(**) are obtained from the conditions of the problem and depend on the co-ordinate system chosen. They must be such that the variation of the independent variable in $[a, b]$ precisely corresponds to the whole of the required magnitude.

III. Evaluation of the definite integral. We have an extremely effective means for doing this — indefinite integration. It becomes evident here that evaluation of the integral, applied to the problem on the basis of its definition (as the limit of an integral sum), is accomplished by an incomparably easier method than is achieved from the actual definition.

On taking the independent variable x as the upper limit of integral(**), we get an expression for $F(x)$ with its known initial value $F(a)$:

$$F(x) = F(a) + \int_a^x f(x) dx.$$

The function $u = F(x)$ is said to give an integral law in $[a, b]$ for the process concerned, its derivative $F'(x) = f(x)$ gives the velocity of the process, and the differential $dF(x) = f(x) dx$ gives its differential law.

Thus our scheme provides a strict basis for the passage from the differential to the integral law.

The first step is usually the most difficult in applications of the integral calculus to practical problems. We have to form here the differential equation of the problem in accordance with the given conditions, i.e. the expression for an arbitrary "element" of the required magnitude. If the differential found is correct, we can assert that the final result obtained in accordance with our scheme is also correct.

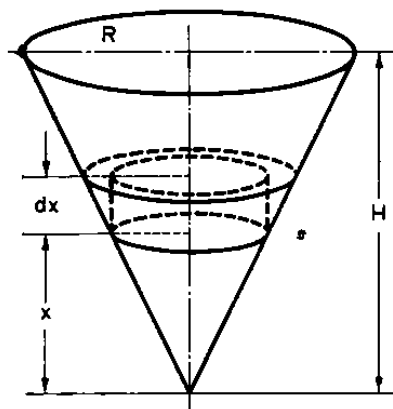


FIG. 126

An example may be mentioned when a careless "simplification" of the increment of the required magnitude leads to an error when finding the differential. Suppose we want to find the lateral surface area of a right circular cone. If we take as the differential of this area the area of the right circular cylinder inscribed in the infinitesimal truncated cone—the increment of the required magnitude over the section $[x, x + dx]$ (Fig. 126)—we arrive at the wrong answer. We advise the reader to work through this problem and indicate the cause of the error.

It will be clear from the problems worked out below (Sec. 116), how it can be verified in concrete cases that the expression, which is usually found with the aid of simplifying assumptions, is in fact the differential of the function.

116. Examples.

I. THE AREA OF A CIRCLE. We take a simple problem, the answer to which is well known. We find the area S of a circle of radius R , knowing that the length of a circumference of radius r is equal to $2\pi r$.

The independent variable in this problem is r , $0 \leq r \leq R$, whilst the known function, "defining" the required quantity, is the length of the circumference $2\pi r$. It is clear that the area of the circle is a function of its radius r . Let us write it as $s = F(r)$; then $S = F(R)$, where S corresponds to the variation of r from 0 to R .

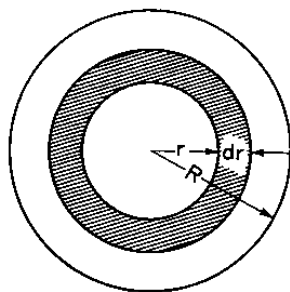


FIG. 127

(1) We form the differential equation of the problem. We assign the increment dr to an arbitrary value r of the radius. The increment of the area Δs is given by the area of the ring shaded in Fig. 127, and we find its principal part ds by supposing that the length of the circumference does not vary over the piece from r to $r + dr$, but remains equal to its length for the "initial value" of the radius r .

The increment (element of the circular area) is easily seen to be equal to the product of the length of the circumference $2\pi r$ and the

increment of the radius dr :

$$ds = 2\pi r dr.$$

We shall verify that the expression $2\pi r dr$ is in fact equivalent to Δs . We obviously have

$$2\pi r dr < \Delta s < 2\pi(r + dr) dr,$$

whence

$$1 < \frac{\Delta s}{2\pi r dr} < 1 + \frac{dr}{r}.$$

These inequalities show that

$$\lim_{dr \rightarrow 0} \frac{\Delta s}{2\pi r dr} = 1.$$

Thus the differential of the area of the circle is equal to the length of the circumference multiplied by the differential of the radius.

(2) We "sum" all the elements when r varies from 0 to R :

$$S = \int_0^R 2\pi r dr.$$

(3) Finally, we evaluate the integral.

The circular area can be imagined very clearly (though in principle incorrectly!) as consisting of a very large number of concentric rings, so that the area of each can be reckoned equal to the length of the inner circumference times a height equal to the width of the ring. This is reflected in the structure of the integrand expression: $2\pi r dr$. The "sum" of these areas (in fact the integral!) gives the area of the total circle.

The integral in this problem expresses an area, though not at all a "trapezoid"; we integrate the length of a variable circumference instead of the ordinate of a curve bounding a trapezoid. Even as applied to problems on areas, the integral is seen to be capable of receiving a treatment entirely different from the original.

Naturally, the problem of the area of a circle can be solved by another method, i.e. finding the area of the corresponding curvilinear trapezoid. We find on taking the equation of the circle as $x^2 + y^2 = R^2$:

$$\begin{aligned} S &= 2 \int_{-R}^R \sqrt{R^2 - x^2} dx = 4 \int_0^R \sqrt{R^2 - x^2} dx = 4R^2 \int_0^1 \sqrt{1 - z^2} dz \\ &= 4R^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \pi R^2. \end{aligned}$$

Problems can be solved by different methods with the aid of integrals by taking different choices of independent variable. Use should be made of this possibility for simplifying the solution.

II. FLOW OF FLUID FROM A CYLINDER. A cylindrical vessel with a hole in the bottom is filled with fluid. Let us find the time T required for the fluid to flow out, if its initial height is H , the cylinder radius is r , and the area of the hole is s (Fig. 128).

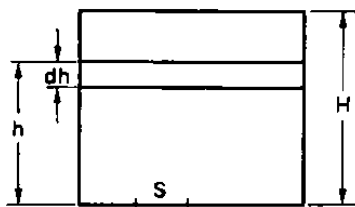


FIG. 128

This problem can be solved with the aid of an integral if use is made of the following law of Torricelli: the speed of fluid flow through a hole which is small compared with the height of the fluid column is equal to $\sqrt{2gh}$, where h (cm) is the height of the fluid level above the hole, and $g = 981$ (as we know, the velocity of free fall of a body in space is similarly expressed as a function of the height of fall). The flow velocity of the fluid is numerically equal to the fluid weight flowing out per unit time (1 sec) through a hole of unit area (1 cm^2).

If the fluid were constantly replenished, the flow velocity would remain constant by Torricelli's law, and the amount of fluid initially present would flow away in $\gamma H \pi r^2 / \sqrt{2gHs}$ units of time (seconds), where γ is the specific weight of the fluid. But without such replenishment the fluid level will fall constantly, and the flow velocity will diminish; thus the problem turns out to be more complicated.

Let the level h (the height of the fluid column above the hole) correspond to the instant t (the level has dropped by $H - h$ after time t).

(1) We form the differential equation of the problem. We assign a negative increment dh to the variable h ; the variable t now receives a (positive) increment Δt (the level drops $-dh$ in time Δt). We find dt as the number of units of time which would be required for the same drop in level if the flow velocity (the "defining" magnitude

in this problem) remained invariable, at its value at the level h (or what amounts to the same thing, at the instant t).

The amount of fluid flowing out in the time interval from t to $t + dt$ is $\sqrt{2ghs} dt$ with our assumption. But on the other hand, it is precisely equal to the amount of fluid contained between the levels h and $h + dh$, i.e. $-\pi r^2 dh$ (the minus sign is taken because $dh < 0$).

We thus have:

$$\sqrt{2ghs} dt = -\gamma \pi r^2 dh.$$

This gives us an expression for the differential of one magnitude in terms of the differential of the other:

$$dt = -\frac{\gamma \pi r^2}{\sqrt{2gs}} \frac{dh}{\sqrt{h}}.$$

(2) The total time of flow T corresponds to variation of h from H to 0. In view of this, "summation of elements" of the required quantity (T) gives

$$T = -\int_H^0 \frac{\pi \gamma r^2}{\sqrt{2gs}} \frac{dh}{\sqrt{h}}.$$

(3) Integration leads to the result:

$$T = \frac{\gamma \pi r^2}{\sqrt{2gs}} \int_0^H h^{-\frac{1}{2}} dh = \frac{\gamma \pi r^2}{\sqrt{2gs}} \frac{H^{1/2} - 0^{1/2}}{\frac{1}{2}} = \frac{2\gamma \pi r^2}{\sqrt{2gs}} \sqrt{H}.$$

This formula gives the time taken for fluid at a given initial level to flow out, or conversely, the initial fluid height for a given total time of flow. It is of interest that the time actually needed for the fluid to flow out is twice as great as that needed if the fluid is constantly replenished.

III. ATTRACTION OF A PARTICLE BY A ROD. We find the force p with which a homogeneous straight rod of length a and density μ attracts a particle M of mass m situated at a distance l from the rod on its perpendicular bisector (Fig. 129).

We mark out two equal segments ON and ON_1 on either side of the mid-point O of the rod. The force p with which a particle M is attracted by the segment N_1N is obviously a function of its length, or what amounts to the same thing, of the angle φ between the segment MN and the perpendicular MO :

$$p = F(\varphi).$$

We assign an increment $d\varphi$ to the independent variable φ . Then the function $F(\varphi)$ receives an increment Δp , this being the force with which the particle is attracted by two equal segments NN' and $N_1N'_1$ symmetrical about O .

We consider first the action on the particle of the segment NN' . To find the differential of the force, we assume that the total mass of NN' is concentrated at the point N . By Newton's law, the attractive force is equal to $km\mu NN'/MN^2$, where k is a constant coefficient. We now remark that $MN = l/\cos \varphi$, where $l = MO$, and

$$NN' = ON' - ON = l[\tan(\varphi + d\varphi) - \tan \varphi] \sim l d \tan \varphi = l \frac{d\varphi}{\cos^2 \varphi}$$

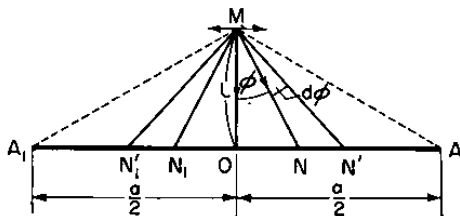


FIG. 129

(we have neglected infinitesimals of higher order than $d\varphi$). Thus the attractive force between NN' and the particle M is equivalent to the force

$$k \frac{m\mu}{l} d\varphi.$$

This force can be imagined as applied to the particle M along the segment MN . We only need to take into account its component in the direction MO , since the second component along the perpendicular to MO is cancelled by the corresponding component of the force on M due to $N_1N'_1$. We therefore obtain, on taking the influence of both segments NN' and $N_1N'_1$ into account:

$$dp = 2k \frac{m\mu}{l} \cos \varphi d\varphi.$$

The correctness of this expression for the differential is obvious, since we have only ignored higher order infinitesimals in composing it.

We "sum" all the elements of the required force; the angle φ has to vary from 0 to the angle between MA and MO , i.e. $\arctan a/2l$. We get:

$$p = 2k \frac{m\mu}{l} \int_0^{\arctan a/2l} \cos \varphi d\varphi = 2k \frac{m\mu}{l} \sin \varphi \Big|_0^{\arctan a/2l} = 2k \frac{m\mu}{l} \sin \arctan \frac{a}{2l}.$$

Bearing in mind that $\sin \alpha = \tan \alpha / \sqrt{1 + \tan^2 \alpha}$, we finally have

$$p = 2k \frac{m\mu}{l} \frac{a}{\sqrt{a^2 + 4l^2}}.$$

Suppose that the length a of the rod increases indefinitely. Then the force p increases, tending to

$$p_{\infty} = 2k \frac{m\mu}{l}.$$

This is the force with which an "infinite" homogeneous straight rod (of density μ) attracts a particle m at a distance l from it. Whilst an infinite rod does not exist in practice, if the rod length is large compared with the distance of the particle from it (on its perpendicular bisector), we can assume approximately that the attractive force is measured by the above very simple expression.

If the rod length $x = ON$ is taken as independent variable instead of the angle φ , the required force is given by the integral

$$p = 2k m \mu l \int_0^{a/2} \frac{dx}{(l^2 + x^2)\sqrt{l^2 + x^2}}.$$

This can be conveniently evaluated by the substitution $x = l \tan \varphi$ (see Secs. 98,9). Now,

$$p = 2k \frac{m\mu}{l} \int_0^{\arctan a/2l} \cos \varphi d\varphi.$$

Hence the choice of the angle φ as independent variable in fact accomplishes the substitution reducing the complicated integral to a tabulated form (of $\cos \varphi$).

The passage from one independent variable to another in a concrete problem solved with the aid of an integral implies no more than a change of variable in the integral. In view of this—let us emphasize it again—a happy choice of independent variable (or what amounts to the same thing, of co-ordinate system) saves us from complicated operations later in finding the integral.

The above problems strikingly illustrate the variety of problems which can be investigated with the aid of the integral.

2. Some Problems of Geometry and Statics. Processes of Organic Growth

117. Area of a figure. All the problems discussed in the present article may be solved afresh each time in accordance with the scheme established in Sec. 115. However, since they are very important and often encountered, we shall solve them here in the general form, and obtain final formulae which we advise the reader to memorize.

We start with the problem of calculating the area of a plane figure, or of the quadrature of a figure, as it is sometimes described. This problem has such a direct connection historically with integration

that the evaluation of an integral may in fact be called a quadrature, and if a problem leads to evaluation of an integral, we may say that it leads to a *quadrature*.

I. CARTESIAN CO-ORDINATE SYSTEM. When specifying the curve bounding the figure by an equation in a Cartesian co-ordinate system, it is convenient to take a curvilinear trapezoid as the basic figure whose area is given by an integral. We have for its area S (on the assumption say that $y \geq 0$):

$$S = \int_{X_1}^{X_2} f(x) dx,$$

where $y = f(x)$ is the equation of the curve bounding the trapezoid, and X_1, X_2 are the values of x corresponding to the beginning and end of this curve (measuring from left to right).

If the equation of the curve is given in the parametric form $x = \varphi(t)$, $y = \psi(t)$, substitution in the integral in accordance with the formula $x = \varphi(t)$ obviously gives us

$$S = \int_{T_1}^{T_2} \psi(t) \varphi'(t) dt,$$

where T_1, T_2 are the values between which t varies when the point runs from left to right over the entire curve bounding the trapezoid.

Examples. (1) Let us find the area S bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. We have:

$$S = 2 \int_{-a}^a y dx.$$

It is convenient here to pass to the parametric form: $x = a \cos t$, $y = b \sin t$. Then

$$S = -2ab \int_{\pi}^0 \sin^2 t dt.$$

Evaluation gives

$$S = 2ab \int_0^{\pi} \sin^2 t dt = 2ab \cdot \frac{1}{2} (t - \sin t \cos t) \Big|_0^{\pi} = \pi ab.$$

The use of the parametric equations amounts here in essence to a trigonometric substitution in the integral

$$2 \frac{b}{a} \int_a^a \sqrt{a^2 - x^2} dx.$$

(2) Let us find the area S of the curvilinear trapezoid bounded by the semi-transverse axis $[0, 1]$ of the rectangular hyperbola $x^2 - y^2 = 1$, the segment of straight line joining the centre $(0, 0)$ to the given point (x, y) of the hyperbola—let this point lie in the first quadrant—and the hyperbola itself. We evidently have

$$S = \frac{1}{2}xy - \int_1^x \sqrt{x^2 - 1} dx.$$

Evaluation of the integral (see Sec. 98, example 8) gives

$$S = \frac{1}{2}xy - \frac{1}{2}[x\sqrt{x^2 - 1} - \ln(x + \sqrt{x^2 - 1})] = \frac{1}{2}\ln(x + \sqrt{x^2 - 1}).$$

Hence we find for x :

$$x = \frac{e^{2S} + e^{-2S}}{2} = \cosh 2S;$$

on writing t for $2S$, we arrive at the first of the parametric equations of the hyperbola given in Sec. 55 (III, example 3), i.e. $x = \cosh t$; we get the second from the equation $\cosh^2 t - y^2 = 1$, whence $y = \sinh t$. We have therefore proved the statement of Sec. 55 about the geometrical meaning of the parameter t in the equations of the hyperbola.

II. POLAR CO-ORDINATE SYSTEM. If the curve bounding the figure is given by an equation in a polar co-ordinate system, a curvilinear sector is taken as the basic figure (Fig. 130).

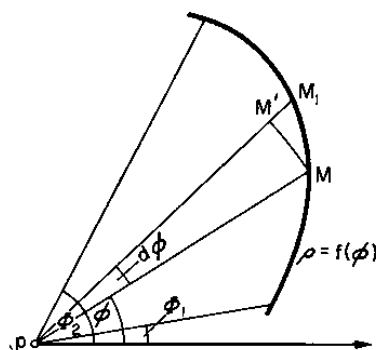


FIG. 130

Definition. A curvilinear sector is a figure bounded by a curve $\rho = f(\phi)$ which is cut by any straight line from the pole P in not more than one point, and by the two radius vectors $\phi = \phi_1$ and $\phi = \phi_2$ of this curve.

Thus $f(\varphi)$ is a single-valued function in the interval $[\Phi_1, \Phi_2]$. We show that the area of the sector is given by an integral.

We take an arbitrary value of φ , $\Phi_1 < \varphi < \Phi_2$, and the corresponding value of ρ . We assign the increment $d\varphi$ to φ ; the area S of the sector now receives the increment ΔS , expressed by the area of sector $PM M_1$. The differential dS of the area may be found by supposing that the radius vector ρ remains constant between φ and $\varphi + d\varphi$. The increment of the area is now given by the area of the circular sector $PM M'$. As we know from elementary geometry, the area of the latter is half the radius times the arc length, i.e.

$$dS = \frac{1}{2} \rho \rho d\varphi = \frac{1}{2} \rho^2 d\varphi.$$

It is easily verified that this expression is in fact the principal part of the increment ΔS .

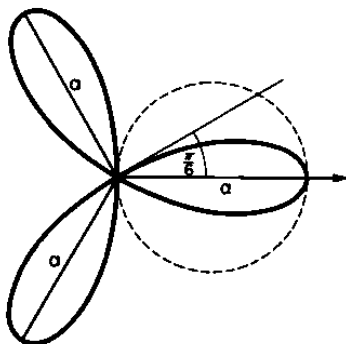


FIG. 131

Summation between the limits Φ_1 and Φ_2 gives

$$S = \frac{1}{2} \int_{\Phi_1}^{\Phi_2} \rho^2 d\varphi.$$

Example. Let us find the area S of the figure bounded by the curve

$$\rho = a \cos 3\varphi,$$

which is known as a three-petal rose (Fig. 131).

The area of one sixth of the figure (half a petal) is given by

$$\frac{1}{2} a^2 \int_0^{\pi/6} \cos^2 3\varphi d\varphi.$$

For, the end of the radius vector describes the whole of the curve bounding the sector when the polar angle varies from 0 to $\pi/6$.

Thus the area of the rose is

$$S = 3a^2 \int_0^{\pi/6} \cos^2 3\varphi \, d\varphi = a^2 \int_0^{\pi/2} \cos^2 u \, du = a^2 \frac{\pi}{4},$$

i.e. the area of a circle of diameter a .

118. Length of arc. The concept of length of arc, like that of the area of a curvilinear figure, requires a special definition.

The length of a straight line segment is established on the basis of the actual process of measurement, which consists, as we know, in fitting into the given segment a segment, or part of it, taken as the unit of length. The number obtained as a result of this comparison in fact measures the length of the segment. Such a measuring process is unrealizable in the case of a curved line, since, generally speaking, no part of a curved line is compatible with a straight line; we therefore have to indicate some "refinement" of the measuring process whereby a number, taken as a measure of the length, can be associated with the given curve.

Just as when defining an area, we replace the curve by different curves which represent it as well as possible and which have easily found lengths. The step-line is a suitable curve. As the step-line indefinitely approaches the given curve, a natural and readily visualized limiting process is obtained for defining the arc length.

We inscribe a step-line in the curve, i.e. a line consisting of pieces of straight lines, with end-points on the curve. The length of the step-line is found by elementary means; starting from obvious geometrical considerations, we take it that, the greater the number of "sides" of the step-line and the smaller the longest of these, the better its length expresses the magnitude which is to be reckoned the length of the given curve. If it is possible to describe the step-line so that each of its sides touches the given curve, then precisely as before, the greater the number of sides and the smaller the longest of them, the more closely its length will approach the magnitude which must inevitably be taken as the length of the curve.

Definition. The *length of an arc (curve)* is the limit to which the inscribed or circumscribed step-line tends on indefinite increase in the number of its sides and indefinite decrease of the length of the greatest side.

(It can be shown that the limits of the perimeters of the inscribed and circumscribed step-lines are the same.)

Not every curve has a length in accordance with this definition. But we are concerned, generally in pure mathematics, and solely in applied mathematics, with curves which have a length, i.e. are *rectifiable*. Evaluation of the length of a curve is called *rectification*.

If the arc is convex, its length in accordance with our definition is easily seen to be greater than the length of any step-line inscribed in it, and less than the length of any step-line circumscribed about it.

This is in fact the principle from which we started when deducing the formula for the differential of an arc in Sec. 57; if the curve is given in a Cartesian co-ordinate system by the equation $y = f(x)$, then

$$ds = \sqrt{1 + f'^2(x)} dx.$$

Assuming that $f'(x)$ is a continuous function and that, in addition, length has the property of additiveness, we obtain by the scheme of Sec. 115 for the total length L of the curve*:

$$L = \int_{X_1}^{X_2} \sqrt{1 + f'^2(x)} dx;$$

here X_1 and X_2 are the abscissae of the left- and right-hand ends respectively of the curve.

* We arrive at the same expression if we start directly from the definition of length. We divide the arc into n parts, and let the points of division have abscissae $X_1 = x_0, x_1, x_2, \dots, x_{n-1}, x_n = X_2$. On drawing the chord through each pair of consecutive points, we obtain an inscribed step-line, the length s_n of which is evidently

$$s_n = \sqrt{\Delta x_0^2 + \Delta y_0^2} + \sqrt{\Delta x_1^2 + \Delta y_1^2} + \dots + \sqrt{\Delta x_{n-1}^2 + \Delta y_{n-1}^2} = \sum_{i=0}^{n-1} \sqrt{\Delta x_i^2 + \Delta y_i^2},$$

or

$$s_n = \sum_{i=0}^{n-1} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i,$$

where $\Delta x_i = x_{i+1} - x_i$, $\Delta y_i = f(x_{i+1}) - f(x_i)$. We have by Lagrange's formula:

$$s_n = \sum_{i=0}^{n-1} \sqrt{1 + f'^2(\xi_i)} \Delta x_i,$$

and by passing to the limit as $n \rightarrow \infty$ and the greatest side tends to zero, we find the formula given in the text for the total length L of the curve. The same result is got if we take the circumscribed instead of the inscribed step-line.

This approach does not require that the curve be "locally convex" at every point (see Sec. 57).

As we know, the differential of the arc can be given a form which does not presuppose the choice of independent variable (variable of integration):

$$ds = \sqrt{dx^2 + dy^2}.$$

We now obtain the general formula for the length of arc in Cartesian co-ordinates:

$$L = \int_{(A)}^{(B)} \sqrt{dx^2 + dy^2}$$

where (A) and (B) denote conventionally the beginning and end of the rectified arc AB . For (A) and (B) we have to substitute the corresponding values of the chosen independent variable.

If the equation of the curve is given parametrically: $x = \varphi(t)$, $y = \psi(t)$, then

$$L = \int_{T_1}^{T_2} \sqrt{\varphi'^2(t) + \psi'^2(t)} dt, \quad T_1 < T_2, \quad (\dagger)$$

where T_1, T_2 are the values of parameter t corresponding to the beginning A and end B of the arc.

On passing to polar co-ordinates, we have the general formula

$$L = \int_{(A)}^{(B)} \sqrt{(\rho d\varphi)^2 + (d\rho)^2}.$$

The variable of integration is very often the polar angle φ (the equation is solved for ρ), and the formula becomes

$$L = \int_{\Phi_1}^{\Phi_2} \sqrt{\rho^2 + \rho'^2} d\varphi, \quad \Phi_1 < \Phi_2.$$

Examples. (1) Let us rectify one arc of the cycloid:

$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$

We have

$$L = a \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + \sin^2 t} dt = 2a \int_0^{2\pi} \sin \frac{t}{2} dt = 4a \int_0^{\pi} \sin u du = 8a,$$

i.e. the length is eight times the radius of the generating circle. This result was obtained in Sec. 83, III from indirect arguments.

(2) For the length of arc of the ellipse

$$x = a \cos t, \quad y = b \sin t$$

we find:

$$L = \int_{T_1}^{T_2} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} dt = a \int_{T_1}^{T_2} \sqrt{1 - \varepsilon^2 \cos^2 t} dt,$$

where $\varepsilon = \sqrt{\frac{a^2 - b^2}{a^2}}$ is the eccentricity. Putting $\cos t = u$, the integral takes another form:

$$L = a \int_{u_1}^{u_2} \frac{1 - \varepsilon^2 u^2}{\sqrt{(1 - u^2)(1 - \varepsilon^2 u^2)}} du, \quad u_1 = \cos T_1, \quad u_2 = \cos T_2.$$

This integral (called an elliptic integral) cannot be found in a closed form, so that the length of arc of the ellipse cannot be expressed by elementary functions. The length can be found approximately by the methods of approximate evaluation of definite integrals (or directly from tables of the values of elliptic integrals).

(3) Let us find the length of the logarithmic spiral $\varrho = ae^{m\varphi}$ from the point (ϱ_0, φ_0) to the variable point (ϱ, φ) .

We have

$$\begin{aligned} L &= \int_{\varphi_0}^{\varphi} \sqrt{a^2 e^{2m\varphi} + a^2 m^2 e^{2m\varphi}} d\varphi = a \sqrt{1 + m^2} \int_{\varphi_0}^{\varphi} e^{m\varphi} d\varphi \\ &= a \frac{\sqrt{1 + m^2}}{m} (e^{m\varphi} - e^{m\varphi_0}). \end{aligned}$$

Here L is given as a function of the final polar angle φ . It can be written as a function of the final radius vector ϱ :

$$L = \frac{\sqrt{1 + m^2}}{m} (\varrho - \varrho_0).$$

This expression shows that the length of the logarithmic spiral is proportional to the increment of the radius vector.

If we move along the spiral from the pole, the polar angle φ tends to $-\infty$. The integral for the length becomes improper, and we get:

$$L = a \sqrt{1 + m^2} \int_{-\infty}^{\varphi} e^{m\varphi} d\varphi = a \frac{\sqrt{1 + m^2}}{m} e^{m\varphi} = \frac{\sqrt{1 + m^2}}{m} \varrho,$$

i.e. the length of the logarithmic spiral from the pole to an arbitrary point is proportional to the radius vector of the point.

Of course there is nothing surprising in the fact that a finite length can be assigned to a curve having an infinite set of turns. The rectifiability of the spiral can be visualized from the simple fact that any finite arc can be used in theory to form a curve with an infinity of turns, the total length of the turns being the length of the original arc. For instance, a straight segment of unit length can be twisted to form an infinite rectangular spiral by repeatedly bending at a right-angle half the previous segment. Its length is equal to 1 ($= 1/2 + 1/4 + 1/8 + \dots$).

119. Volume of a body. We first of all lay down the two visual principles which we take as our starting-point in defining the volume of a solid:

(1) *the volume of a solid has the property of additiveness, i.e. the total volume is equal to the sum of the volumes of the parts from which the body is composed;*

(2) *the volume of a right cylinder is equal to the area of the figure contained in the parallel bases of the cylinder multiplied by its height.*

Suppose we have a solid bounded by a closed surface and that we know the area of the section of it by any plane perpendicular to a given straight line, say Ox (Fig. 132).

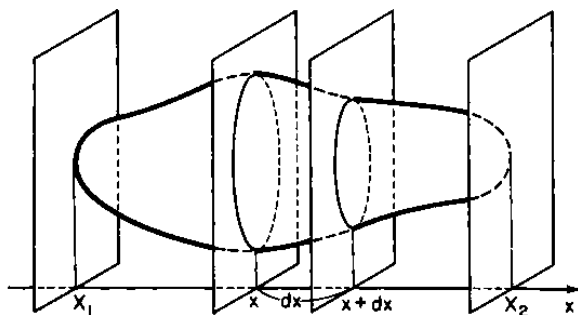


FIG. 132

We can assume that the sectional area is a known function $S(x)$, where x is the abscissa of the point of intersection of the cutting plane with Ox .

Suppose, further, that the entire solid is contained between two planes perpendicular to Ox , cutting Ox at X_1 and X_2 .

Our aim is to find an expression for the volume V of the solid. To do this, we draw an arbitrary plane perpendicular to Ox and cutting it at the point x , $X_1 < x < X_2$; we consider the volume v

of the body measured say from abscissa X_1 to the abscissa x ; we assign x the increment Δx ; then v receives the increment Δv , expressed by the volume of the part of the body lying between the planes through x and $x + dx$ (Fig. 132). We take instead of this part the right cylinder whose base is the section with abscissa x and whose height is dx . The increment of the volume is now given by $S(x) dx$. This is the principal part of the increment Δv . For, clearly, Δv is subject to the inequalities

$$\underline{S} dx \leq \Delta v \leq \bar{S} dx,$$

where $\underline{S}$ and $\bar{S}$ denote respectively the least and greatest areas of the sections in the interval $[x, x + dx]$. On dividing the inequalities through by $S(x) dx$, we get:

$$\frac{\underline{S}}{\bar{S}} \leq \frac{\Delta v}{S(x) dx} \leq \frac{\bar{S}}{\underline{S}}.$$

We assume that $S(x)$ is a continuous function; hence as $dx \rightarrow 0$, $\underline{S} \rightarrow S(x)$ and $\bar{S} \rightarrow S(x)$. Thus

$$\frac{\Delta v}{S(x) dx} \rightarrow 1.$$

This is what we needed to prove.

Hence

$$dv = S(x) dx.$$

Integration gives:

$$v = \int_{X_1}^{X_2} S(x) dx \quad (X_1 < X_2). \quad (\dagger)$$

Since finding the expression for $S(x)$ in general also requires integration, we can say that evaluation of the volume of a solid leads to two quadratures.

If the body is obtained by revolution of the curve $y = f(x)$ about Ox (it being naturally assumed that the curve does not cut Ox in the interval $[X_1, X_2]$), the sectional area with abscissa x is a circle, whose radius is equal to the corresponding ordinate of the curve $y = f(x)$.

In this case

$$S(x) = \pi y^2$$

and we arrive at the formula for the volume of a solid of revolution

$$V = \pi \int_{X_1}^{X_2} f^2(x) dx \quad (X_1 < X_2).$$

A simple and convenient rule may be mentioned for calculating the volumes of solids in a particular case. We assume that the area $S(x)$ of the section of the solid is given by a second degree polynomial in x , i.e. $S(x) = ax^2 + bx + c$. Then we have (see Sec. 108, III):

$$V = \int_{x_1}^{x_2} S(x) dx = \frac{H}{6} (S_H + 4S_C + S_K),$$

where $H = X_2 - X_1$ is the height of the solid, $S_H = S(X_1)$, $S_C = \frac{1}{2}S(X_1 + X_2)$, $S_K = S(X_2)$ are the areas of the initial, central, and final sections of the solid. This is known as Simpson's formula, though it was previously known to Cavalieri (1591–1647). The formula is quite widely applicable. For instance, its correctness is easily verified for solids whose volumes are known from elementary geometry: the prism, pyramid, sphere, cylinder, cone (entire and truncated). If the variation of the cross-sectional area is not subject to the required law, the formula is used as an approximation. (It can be shown that it still remains accurate if $S(x)$ is a third degree polynomial in x : $S(x) = ax^3 + bx^2 + cx + d$.) Actually, this use of the formula implies approximate evaluation of integral (†) by Simpson's rule (see Sec. 108) with one intermediate point of division.

Cavalieri's principle follows directly from the general formula (†).

Two solids bounded by parallel planes have equal volumes if their heights are equal and the areas of sections parallel to these planes, and at the same distance from them, are equal.

For, in view of the hypotheses, the solid can be displaced so that the limits of integrals (†) and the integrands are identical in both cases.

Cavalieri's principle lost its importance after the creation of the differential and integral calculus, but it had a great influence in its time on the development of the ideas lying at the basis of mathematical analysis.

Examples. (1) The plane section of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

say perpendicular to Ox (Fig. 133) is an ellipse with semi-axes $b\sqrt{1 - (x^2/a^2)}$ and $c\sqrt{1 - (x^2/a^2)}$, $-a \leq x \leq a$.

Let us find the volume of the ellipsoid.

The cross-sectional area at the point x is familiar to us (see Sec. 117):

$$S(x) = \pi b \sqrt{1 - \frac{x^2}{a^2}} c \sqrt{1 - \frac{x^2}{a^2}} = \pi b c \left(1 - \frac{x^2}{a^2}\right);$$

it is seen to be given by a quadratic function in x . Simpson's rule is therefore applicable. Here,

$$H = 2a, S_H = S_K = 0, S_C = \pi b c,$$

so that

$$V = \frac{2a}{6} 4\pi b c = \frac{4}{3} \pi a b c.$$

If two of the semi-axes are equal, say $c = b$, we obtain for the volume of the ellipsoid of revolution: $V = 4\pi ab^2/3$. If all three semi-axes are equal, $a = b = c$, we get the volume of a sphere: $V = 4\pi a^3/3$.

(2) Let us find the volume of a torus—the body obtained by revolution of a circle about an axis in the same plane; the axis does not cut the circle. Let the circle of radius a revolve about Ox , its centre being at the point $(0, b)$, $b > a$ (Fig. 134). The equation of the revolving circle will be

$$x^2 + (y - b)^2 = a^2.$$

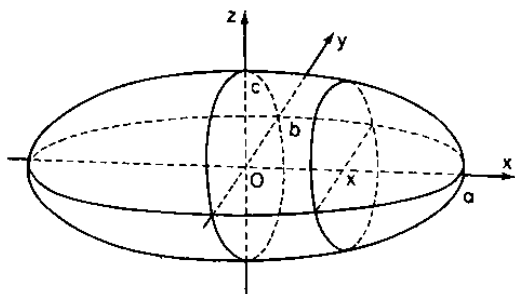


FIG. 133

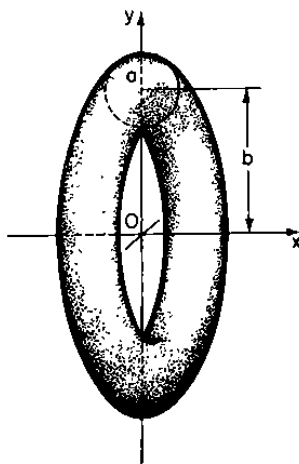


FIG. 134

The required volume is evidently found if the volume of the body formed by revolution of the lower half of the circle ($y = b - \sqrt{a^2 - x^2}$) is subtracted from the volume formed by revolution of the upper half ($y = b + \sqrt{a^2 - x^2}$). We have:

$$\begin{aligned} V &= 2\pi \int_0^a (b + \sqrt{a^2 - x^2})^2 dx - 2\pi \int_0^a (b - \sqrt{a^2 - x^2})^2 dx \\ &= 2\pi \int_0^a [(b + \sqrt{a^2 - x^2})^2 - (b - \sqrt{a^2 - x^2})^2] dx = 8\pi b \int_0^a \sqrt{a^2 - x^2} dx, \end{aligned}$$

and further, on putting $x = a \sin t$:

$$V = 8\pi b a^2 \int_0^{\pi/2} \cos^2 t \, dt = 8\pi b a^2 \frac{\pi}{4} = 2\pi^2 a^2 b.$$

(3) A conoid is a solid whose bases are two plane figures (or curves) lying in parallel planes, and whose lateral surface is composed from the straight lines joining points of the base contours in accordance with some law.

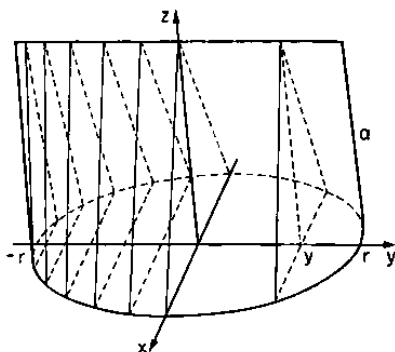


FIG. 135

Let us find the volume V of the following conoid: the lower base is a circle of radius r , the upper a straight segment of length $2r$, located above a diameter of the circle at a height a ; points of the circle and segment having a common projection on this diameter are joined by straight lines (Fig. 135). The best choice of co-ordinate system is clear from the figure. The section perpendicular to the axis of ordinates at the point y is an isosceles triangle, of base $2\sqrt{r^2 - y^2}$, and height a . Thus

$$S(y) = a \sqrt{r^2 - y^2}$$

and

$$V = a \int_{-r}^r \sqrt{r^2 - y^2} \, dy = 2a \int_0^r \sqrt{r^2 - y^2} \, dy = 2a r^2 \int_0^{\pi/2} \cos^2 t \, dt = \frac{1}{2} \pi r^2 a,$$

i.e. the volume of the conoid is equal to half the volume of the right cylinder with the same base and height.

120. Area of surface of revolution. We shall at present just consider the problem of finding the area of a surface of revolution (for the general case of surface area see Part II, Sec. 178).

Let the arc AB of a curve $y = f(x)$ revolve about Ox (Fig. 136), forming a surface, the area Q of which we want to find. It will be assumed that arc AB is "locally convex" and always has a tangent.

We shall start from two natural principles:

- (1) *the surface area has the property of additiveness;*
- (2) *if the tangent and chord are drawn to a revolving infinitesimal convex (concave) arc, the surface area is less than (greater than) the surface area obtained by revolution of the corresponding segment of tangent and is greater than (less than) the surface area obtained by revolution of the chord.*

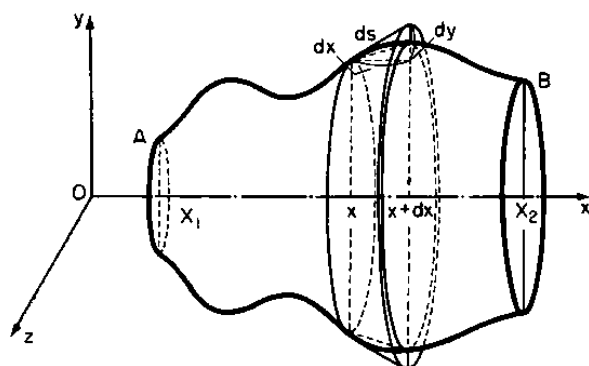


FIG. 136

Let X_1 and X_2 denote the abscissae of the ends of the arc ($X_1 < X_2$). A plane perpendicular to Ox is drawn through an arbitrary point x , $X_1 < x < X_2$. We assign x the increment dx ; then the area q of the surface, reckoned from some fixed section, receives the increment Δq , equal to the area of the surface lying between planes perpendicular to Ox through x and $x + dx$. In accordance with principle (2), Δq satisfies the following inequalities (taking say $dy > \Delta y$):

$$\frac{2\pi y + 2\pi(y + \Delta y)}{2} \sqrt{\Delta x^2 + \Delta y^2} < \Delta q < \frac{2\pi y + 2\pi(y + dy)}{2} \sqrt{dx^2 + dy^2}.$$

For, the expression on the left measures the surface area of the truncated cone whose generator is the chord, whilst the expression on the right gives the surface area of the truncated cone whose generator is the segment of tangent. We find on dividing all the terms of the inequalities by $2\pi y \sqrt{dx^2 + dy^2}$:

$$\frac{4\pi y + 2\pi \Delta y}{4\pi y} \frac{\sqrt{\Delta x^2 + \Delta y^2}}{\sqrt{dx^2 + dy^2}} < \frac{\Delta q}{2\pi y \sqrt{dx^2 + dy^2}} < \frac{4\pi y + 2\pi dy}{4\pi y}.$$

Since both left- and right-hand sides tend to 1 as $dx \rightarrow 0$, we have

$$\frac{\Delta q}{2\pi y \sqrt{dx^2 + dy^2}} \rightarrow 1.$$

The numerator of the fraction—the principal part of Δq —, being proportional to $dx(2\pi y \sqrt{dx^2 + dy^2} = 2\pi y \sqrt{1 + y'^2} dx)$, is the required differential:

$$dq = 2\pi y \sqrt{dx^2 + dy^2} \text{ or more briefly, } dq = 2\pi y ds.$$

Consequently,

$$Q = 2\pi \int_{(A)}^{(B)} f(x) ds.$$

We have to replace (A) and (B) by the corresponding values of the chosen variable of integration. If the equation of the curve is solved for y , the formula conveniently takes the form

$$Q = 2\pi \int_{X_1}^{X_2} f(x) \sqrt{1 + f'^2(x)} dx, \quad X_1 < X_2;$$

whereas if the equation is parametric: $x = \varphi(t)$, $y = \psi(t)$, we have

$$Q = 2\pi \int_{T_1}^{T_2} \psi(t) \sqrt{\varphi'^2(t) + \psi'^2(t)} dt, \quad T_1 < T_2,$$

where T_1 and T_2 are the values of parameter t corresponding to points A and B of the curve.

Examples. (1) Let us find the area of a spherical zone. Let an arc of a circle with centre at the origin and radius r revolve about Ox . We have from the equation of the circle $x^2 + y^2 = r^2$: $y^2 = r^2 - x^2$, $yy' = -x$, i.e.

$$\begin{aligned} Q &= 2\pi \int_{X_1}^{X_2} y \sqrt{1 + y'^2} dx = 2\pi \int_{X_1}^{X_2} \sqrt{y^2 + (yy')^2} dx \\ &= 2\pi \int_{X_1}^{X_2} \sqrt{(r^2 - x^2) + x^2} dx = 2\pi r (X_2 - X_1) = 2\pi r H, \end{aligned}$$

where H is the height of the zone. With $H = 2r$, we get the area of a sphere $Q = 4\pi r^2$.

(2) Let us find the surface area of the torus (see Sec. 119, example 2). It is evidently equal to the sum of the surface areas formed by revolution of the upper and lower halves of the circle $x^2 + (y - b)^2 = a^2$. Since

$$y = b \pm \sqrt{a^2 - x^2} \quad \text{and} \quad y'^2 = \frac{x^2}{a^2 - x^2},$$

we have

$$\begin{aligned} Q &= 2\pi \int_{-a}^a (b + \sqrt{a^2 - x^2}) \sqrt{1 + \frac{x^2}{a^2 - x^2}} dx + \\ &\quad + 2\pi \int_{-a}^a (b - \sqrt{a^2 - x^2}) \sqrt{1 + \frac{x^2}{a^2 - x^2}} dx \\ &= 8\pi a b \int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = 8\pi a b \int_0^1 \frac{du}{\sqrt{1 - u^2}} = 8\pi a b \cdot \arcsin 1 = 4\pi^2 a b. \end{aligned}$$

121. Centre of gravity and Guldin's theorems. If masses $m_1, m_2, \dots, m_n$ are concentrated at points $M_1(x_1, y_1), M_2(x_2, y_2), \dots, M_n(x_n, y_n)$ on a plane, the co-ordinates of the centre of gravity of the system $M(\xi, \eta)$ are given by

$$\xi = \frac{\sum_{i=1}^n x_i m_i}{\sum_{i=1}^n m_i}, \quad \eta = \frac{\sum_{i=1}^n y_i m_i}{\sum_{i=1}^n m_i}.$$

We observe that, for a homogeneous system of particles ($m_1 = m_2 = \dots = m_n$), the co-ordinates of the centre of gravity:

$$\xi = \frac{\sum_{i=1}^n x_i}{n}, \quad \eta = \frac{\sum_{i=1}^n y_i}{n}$$

do not depend on the mass but only on the positions of the particles.

The statical moment of a particle about an axis is the product of its mass and its distance from the axis. Thus the products $x_i m_i$ and $y_i m_i$ are the statical moments of particle M_i about Oy and Ox respectively. The sum of the statical moments of the particles is called the statical moment of the system. Hence the co-ordinates of the centre of gravity of the system can be expressed as

$$\xi = \frac{M_y}{M}, \quad \eta = \frac{M_x}{M}$$

where M_x and M_y are the statical moments of the system about Ox and Oy and M is the total mass. Hence $\xi M = M_y$, $\eta M = M_x$.

The centre of gravity can therefore be defined as the point such that, if the total mass of the system is concentrated at it, its statical moment about an axis is equal to the corresponding statical moment of the entire system.

We now turn to the question of the centres of gravity of continuous distributions (curves, plane figures, surfaces, solids) instead of systems composed

of a finite number of particles, whilst retaining our definition of centre of gravity. The problem amounts to finding the statical moments of such distributions about the co-ordinate axes; this involves again turning to the concept of integral.

I. CURVES. Let us find the statical moments M_x and M_y of a homogeneous material arc AB of the curve $y = f(x)$ (Fig. 137), of density δ ; it will be observed that the curve has only one dimension.

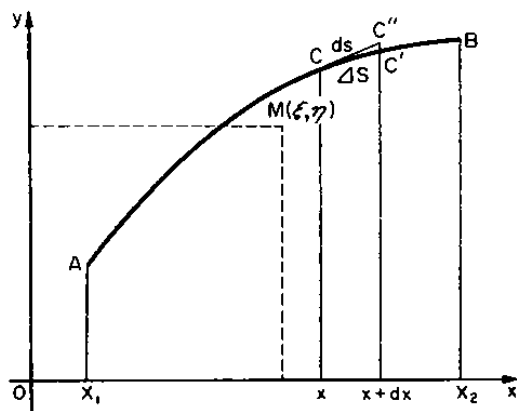


FIG. 137

Let X_1 and X_2 be the abscissae of the ends of the arc. Given any x , $X_1 < x < X_2$, let it receive the increment dx . Then the statical moment m_x of arc AC about Ox receives the increment Δm_x , equal to the statical moment of arc CC' (because the statical moment of the whole is equal to the sum of the statical moments of the parts). We get the principal part of the increment, proportional to dx , i.e. dm_x , if we replace arc CC' by the segment of tangent CC'' and take the statical moment of the point C on the assumption that the total mass of segment CC'' is concentrated at it. Since this mass is obviously equal to the density δ times the segment length ds , we have

$$dm_x = \delta y ds.$$

Similarly,

$$dm_y = \delta x ds.$$

Integration gives us

$$M_x = \delta \int_{(A)}^{(B)} y ds, \quad M_y = \delta \int_{(A)}^{(B)} x ds.$$

We have to replace (A) and (B) by the corresponding values of the chosen variable of integration.

To find the co-ordinates of the centres of gravity of the arc, we simply divide M_y and M_x by the total mass of the arc. Thus

$$\xi = \frac{\int_{(A)}^{(B)} x ds}{\int_{(A)}^{(B)} ds}, \quad \eta = \frac{\int_{(A)}^{(B)} y ds}{\int_{(A)}^{(B)} ds}. \quad (\dagger)$$

It is clear from this that the co-ordinates of the centre of gravity are independent of the density for a homogeneous arc.

We show that, *if the arc has an axis of symmetry, the centre of gravity must lie on it.* We arrange the co-ordinate system so that the axis of symmetry is Oy . We have to show that now $\xi = 0$, i.e. that

$$\int_{(A)}^{(B)} x \, ds = 0.$$

In view of the symmetry of the arc about Oy , we have

$$X_2 = -X_1 = \alpha, \quad f(-x) = f(x), \quad f'(-x) = -f'(x),$$

whence $f'^2(-x) = f'^2(x)$, and the integral

$$\int_{(A)}^{(B)} x \, ds = \int_{-\alpha}^{\alpha} x \sqrt{1 + y'^2} \, dx$$

in fact vanishes, since the integrand is odd in the interval $[-\alpha, \alpha]$.

II. CURVILINEAR TRAPEZOID. We pass on to finding the statical moments and co-ordinates of the centre of gravity of a homogeneous plane material figure (lamina); it is assumed to have only two dimensions (no thickness); the density is δ .

Thanks to the property of the statical moment, that the moment of the whole is equal to the sum of the moments of the parts, the problem amounts to finding the statical moments of curvilinear trapezoids about the co-ordinate axes.

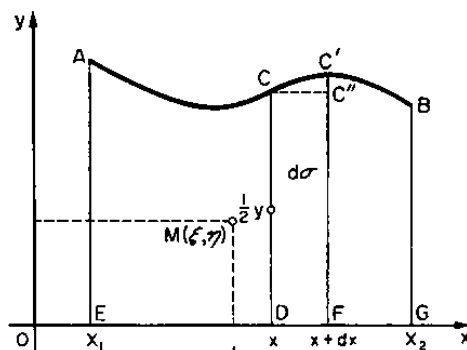


FIG. 138

We take a curvilinear trapezoid with base on Ox , bounded by the arc AB of a curve $y = f(x)$. Given an arbitrary x , $X_1 < x < X_2$ (X_1, X_2 are the abscissae of the ends of the arc), we assign it the increment dx . The statical moments m_x and m_y of trapezoid $CC'FD$ (Fig. 138) now receive the increments Δm_x and Δm_y , these being the moments of the trapezoid $CC'FD$ about Ox and Oy . We find the principal parts of the increments, proportional to dx (i.e. dm_x and dm_y) by replacing this trapezoid by the rectangle $CC''FD$ and taking the statical moments of the ordinate CD on the assumption that the total mass of rectangle $CC''FD$ is uniformly concentrated on it. This mass is equal to the

density δ times the area $d\sigma$ of the rectangle; as regards the moments of CD , these are equal to the moments of its centre of gravity, which obviously lies at the mid-point of the segment. Thus

$$dm_x = \frac{1}{2} \delta y d\sigma, \quad dm_y = \delta x d\sigma,$$

whence

$$M_x = \frac{1}{2} \delta \int_{(A)}^{(B)} y d\sigma, \quad M_y = \delta \int_{(A)}^{(B)} x d\sigma.$$

Since the mass of the total trapezoid is equal to $\delta \int_{(A)}^{(B)} d\sigma$, the formulae for the co-ordinates of the centre of gravity become

$$\xi = \frac{\int_{(A)}^{(B)} x d\sigma}{\int_{(A)}^{(B)} d\sigma}, \quad \eta = \frac{1}{2} \frac{\int_{(A)}^{(B)} y d\sigma}{\int_{(A)}^{(B)} d\sigma}.$$

We have to replace (A) and (B) by the corresponding values of the chosen variable of integration. The formulae most commonly take the form

$$\xi = \frac{\int_{x_1}^{x_2} x y dx}{\int_{x_1}^{x_2} y dx}, \quad \eta = \frac{1}{2} \frac{\int_{x_1}^{x_2} y^2 dx}{\int_{x_1}^{x_2} y dx}. \quad (\dagger\dagger)$$

It can be shown here, even more simply than in I, that, if there is an axis of symmetry, the centre of gravity must lie on it.

We see again that, in the homogeneous case, the co-ordinates of the centre of gravity are independent of the density.

In view of the fact that actual evaluation of the co-ordinates of the centres of gravity of curves and plane figures does not differ from similar evaluations in problems of a purely geometrical kind, we shall not analyze concrete examples, but merely dwell on one very interesting geometrical corollary, consisting in a simple rule for finding the surface areas and volumes of solids of revolution.

III. GULDIN'S THEOREMS. We have from the second of formulae($\dagger$):

$$L \cdot \eta = \int_{(A)}^{(B)} y ds,$$

where L is the length of arc AB (Fig. 136). On multiplying both sides by 2π , we get

$$L \cdot 2\pi \eta = 2\pi \int_{(A)}^{(B)} y ds.$$

The right-hand side gives the surface area formed by revolution of arc AB about Ox (Sec. 120), whilst the factor $2\pi\eta$ on the left-hand side is the length of the circumference described by the centre of gravity of the arc.

Guldin's first theorem*. *The surface area obtained by revolution of an arc about an axis not intersecting it is equal to the length of the arc multiplied by the length of the path described by the centre of gravity of the arc (on the assumption that it is uniform):*

$$Q = Ll, \text{ where } l = 2\pi\eta.$$

The theorem remains valid if the arc does not make a complete revolution (through 2π), but only revolves through an angle α about the axis ($\alpha < 2\pi$); then $l = \alpha\eta$. It is also true when a closed curve (not intersecting the axis) revolves. This can be proved by comparing the explicit expressions for the centre of gravity of a curve and for the area of the corresponding surface of revolution.

This theorem enables us to find the third, given any two of Q , L , η .

For instance, the surface area of the torus (see Sec. 120, example 2) is obtained directly: $L = 2\pi a$, $l = 2\pi b$, so that

$$Q = 4\pi^2 ab.$$

Let us also find the surface area formed by revolution of a semi-circular arc about the tangent at its mid-point. We want to know the position of the centre of gravity of the semi-circular arc. It is clear that it lies on the radius perpendicular to the diameter joining the ends of the arc. Revolution of the arc about this diameter gives a sphere; by Guldin's theorem

$$4\pi r^2 = \pi r \cdot 2\pi\eta,$$

whence

$$\eta = \frac{2r}{\pi}.$$

Consequently the centre of gravity is at a distance $r - 2r/\pi$ from the axis of revolution.

On again applying Guldin's theorem, we obtain the required area:

$$Q = \pi r \cdot 2\pi \left(r - \frac{2r}{\pi} \right) = 2\pi r^2 (\pi - 2).$$

We pass on to Guldin's second theorem.

We have from the second of formulae(††):

$$S \cdot 2\eta = \int_{x_1}^{x_2} y^2 dx,$$

where S is the area of trapezoid $ABGE$ (Fig. 138); we obtain by multiplying both sides by π :

$$S \cdot 2\pi\eta = \pi \int_{x_1}^{x_2} y^2 dx.$$

The right-hand side gives the volume of the solid formed by revolution of the trapezoid about Ox (Sec. 119).

* P. Guldin (1577–1643). The theorems bearing Guldin's name were known to the Alexandrian scholar Pappus (300 B.C.).

Guldin's second theorem. *The volume of the solid obtained by revolution of a trapezoid about its base is equal to its area multiplied by the length of the path described by the centre of gravity of the trapezoid (assuming it to be homogeneous):*

$$V = Sl, \quad \text{where } l = 2\pi\eta.$$

The theorem still holds if the trapezoid only revolves through an angle α ($\alpha < 2\pi$) instead of making a complete revolution; then $l = \alpha\eta$. It is also true when any other figure (not intersecting the axis) revolves. This can be proved by comparing the explicit expressions for the centre of gravity of the figure and for the volume of the corresponding solid of revolution.

Guldin's second theorem enables the third to be found, given any two of V , S , η .

For instance, the volume of the torus (see Sec. 119, example 2) is obtained directly: $S = \pi a^2$, $l = 2\pi b$, i.e.

$$V = 2\pi^2 a^2 b.$$

Let us also find the volume of the solid formed by revolution of a semi-circular disc about the tangent at the mid-point of the semi-circle. It is necessary to find the centre of gravity. Evidently, it lies on the radius perpendicular to the base of the disc. Revolution of the disc about this base gives a sphere; by Guldin's theorem:

$$\frac{4}{3} \pi r^3 = \frac{1}{2} \pi r^2 \cdot 2\pi\eta$$

whence

$$\eta = \frac{4r}{3\pi}.$$

Thus the centre of gravity is at a distance $r - 4r/3\pi$ from the axis of revolution. Again, we find the required volume by applying Guldin's theorem:

$$v = \frac{1}{2} \pi r^2 \cdot 2\pi \left(r - \frac{4r}{3\pi} \right) = \frac{\pi r^2}{3} (3\pi - 4).$$

REMARK 1. Determination of the statical moments and co-ordinates of the centres of gravity of solids and surfaces is based on considerations precisely similar to those explained above in cases I and II. For instance, it is clear that the centres of gravity of homogeneous solids and surfaces of revolution lie on the axis of revolution. Let the axis of revolution be Ox . Then the statical moment of a solid about Oy has the form

$$M_y = \delta \int_{(A)}^{(B)} x dv$$

where δ is the density, A and B are the ends of the revolving arc, dv is the differential of the volume of the solid. We have for the abscissa of the centre of gravity:

$$\xi = \frac{\int_{(A)}^{(B)} x dv}{\int_{(A)}^{(B)} dv} = \frac{\int_{X_1}^{X_2} x y^2 dx}{\int_{X_1}^{X_2} y^2 dx}, \quad X_1 < X_2,$$

where X_1 , X_2 are the abscissae of A and B .

Similarly, we obtain for the abscissa of the centre of gravity of a surface:

$$\xi = \frac{\int_{x_1}^{x_2} x y \sqrt{1+y'^2} dx}{\int_{x_1}^{x_2} y \sqrt{1+y'^2} dx}, \quad X_1 < X_2.$$

REMARK 2. In addition to statical moments, moments of inertia are used in mechanics. The moment of inertia of a particle about an axis is the mass of the particle multiplied by the square of its distance from the axis. Hence the statical moment is sometimes called a first order moment, or linear moment (it contains the first degree of the distance), whilst the moment of inertia is a second order moment (it contains the second power of the distance.)

We can pass from the moment of inertia of a particle to the moment of inertia of a continuous distribution (curve, figure, surface, solid) in precisely the same way as when dealing with statical moments. It is quite obvious that the expressions obtained for the moments of inertia about the co-ordinate axes will differ from the corresponding expressions for the statical moments only in the appearance of the squares instead of the first powers of the co-ordinates under the integrals. For instance, the moments of inertia I_y about Oy of a curve, trapezoid, surface of revolution and solid of revolution are given respectively by:

$$I_y = \delta \int_{(A)}^{(B)} x^2 ds, \quad I_y = \delta \int_{(A)}^{(B)} x^2 d\sigma, \quad I_y = \delta \int_{(A)}^{(B)} x^2 dq, \quad I_y = \delta \int_{(A)}^{(B)} x^2 dv.$$

122. Processes of organic growth. A process of organic growth is any process in which two variables x and y take part, and the differential law of which is given by

$$dy = ky dx,$$

where k is a constant (see Sec. 58).

Integration leads to an exponential function, with the aid of which the integral law of the process is expressed:

$$y = C_0 e^{kx}.$$

I. REPRODUCTION OF MICRO-ORGANISMS. Suppose we know that a colony of micro-organisms weighs $M_0 = 0.05 g$ at the start of the experiment, and weighs $M_2 = 0.06 g$ after 2 hours; let us find how much the colony will weigh after 3 hours.

We write M for the weight at the instant t ; assuming that the increase in weight, due to reproduction, follows the law of organic growth $dM = kM dt$, we obtain after integration:

$$M = M_0 e^{kt}.$$

Thus $M_2 = 0.06 = 0.05 e^{k \cdot 2}$, whence $k = 0.0915$. The weight is therefore given as a function of time by

$$M = 0.05 e^{0.0915t}.$$

In particular, we find on substituting $t = 3$:

$$M_3 = 0.05 e^{0.0915 \cdot 3} = 0.0658 g.$$

II. DECAY OF RADIUM. We are presented here with an example of a process in which one magnitude decreases with the decrease of another in accordance with a law of organic growth.

Since every part of a given stock of radium must be regarded as a source of the formation of new material, it is clear that, the greater the amount of radium at the initial instant, the greater the amount of substance produced from it. Assuming a law of organic growth, we write:

$$\frac{dx}{dt} = -kx, \quad \text{or} \quad dt = -\frac{1}{k} \frac{dx}{x},$$

where x is the amount of radium at instant t , and k is a constant. The minus sign is taken because positive dt corresponds to negative dx : as the time interval increases, the amount of radium diminishes.

We have

$$t = -\frac{1}{k} \int_a^x \frac{dx}{x},$$

where a is the amount of radium at $t = 0$. Thus

$$t = -\frac{1}{k} \ln \frac{x}{a} \quad \text{and} \quad x = a e^{-kt}.$$

Given a and k , we can find x in terms of t or t in terms of x from these equations.

Suppose that say half the initial amount $a = 1$ g of radium B decays (forming radium C) in the course of 26.7 min. The equation $26.7 = -\frac{1}{k} \ln \frac{0.5}{1}$ gives us $k \approx 0.026$, i.e. the process follows the law

$$x = e^{-0.026t}.$$

We only need to put $t = 10$ in this formula to find how much has not decayed 10 min from the start of the experiment:

$$x = e^{-0.026 \cdot 10} \approx 0.77 \text{ g}.$$

Since e^{-kt} decreases more and more slowly with increasing t , the decay of radium is a process which slows down.

III. BAROMETRIC FORMULA. We consider a prismatic column of air with 1 m^2 cross-section, the base being at sea-level. Obviously, the pressure p at a height h is a decreasing function of h . If the air density did not change with height, the pressure at height h would be less than the pressure at sea-level simply by an amount equal to the weight of an air column of height h , i.e. $k_0 h$, where k_0 is the air density at sea-level. But the density in turn depends on the pressure and also decreases with height.

Let h be given the positive increment dh , and let us find the connection between dh and dp . Corresponding to the increase dh of h we have a decrease Δp in the pressure, numerically equal to the weight of air between two horizontal sections of the column at heights h and $h + dh$. We get dp by supposing that the density is constant in the interval from h to $h + dh$, and equal to the density at h .

Then

$$dp = -k dh,$$

where k is the density at h .

We make the assumption that the air temperature remains constant from sea-level to the height of interest. Then by the Boyle-Mariotte law, we have

$$k = \frac{k_0}{p_0} p,$$

where p_0 is the pressure at sea-level. Thus

$$dp = -\frac{k_0}{p_0} p dh \quad \text{and} \quad dh = -\frac{p_0}{k_0} \frac{dp}{p}.$$

Since the pressure change corresponding to height h is from p_0 to p , we have

$$h = -\frac{p_0}{k_0} \int_{p_0}^p \frac{dp}{p} = -\frac{p_0}{k_0} \ln \frac{p}{p_0}.$$

Hence

$$p = p_0 e^{-k_0 h/p_0}$$

This formula, giving the pressure in terms of the height, is known as the simplified barometric (or hypsometric) formula. The simplification lies in the assumption of constant temperature from sea-level to the height h . This assumption is not valid in nature; the temperature changes fairly rapidly with height; the Boyle-Mariotte law cannot be used in the solution of the problem if this change is to be taken into account. Our formula can only be used as a general guide for fairly small h . The more exact barometric formula is much more complicated.

SERIES

1. Numerical Series

123. Series. Convergence.

I. SERIES. The concept of series makes its appearance even in elementary mathematics, in connection with the problem of expressing a given number in terms of other numbers, as e.g. in terms of decimals, which are the simplest numbers in the ordinary way. Suppose we want to express $5/8$ and $5/9$ as decimals. Division of the numerator by the denominator leaves no remainder in the first case ($5/8$ is equal to the "finite" decimal 0.625), whereas the division does not terminate in the second case ($5/9$ is equal to the "infinite" recurring decimal $0.555 \dots$). The number $5/8$ is thus the sum of decimal fractions: $\frac{6}{10} + \frac{2}{10^2} + \frac{5}{10^3}$, whereas $5/9$ can only be regarded as a sum of decimal fractions if we agree to take into consideration an expression consisting of an "infinite" set of fractions:

$$\frac{5}{10} + \frac{5}{10^2} + \frac{5}{10^3} + \dots + \frac{5}{10^n} + \dots,$$

and ascribe to this a sum worked out in accordance with a definite rule (in this case $5/9 = \frac{5/10}{1 - 1/10}$). We have obtained an example of a series—a geometrical progression—studied in the course of elementary mathematics. This will be the case whenever we write a rational number in terms of decimal fractions, i.e. "convert" an ordinary fraction into decimals (except in cases when the rational number is equal to a finite decimal fraction).

If we try to write an irrational number by means of decimal fractions, we again arrive at an "infinite" fraction, though it is no longer a geometrical progression: the decimal is not recurring. For instance:

$$\sqrt{2} = 1.4142 \dots = 1 + \frac{4}{10} + \frac{1}{10^2} + \frac{4}{10^3} + \frac{2}{10^4} + \dots,$$

$$\pi = 3.1415 \dots = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \dots,$$

$$e = 2.7182 \dots = 2 + \frac{7}{10} + \frac{1}{10^2} + \frac{8}{10^3} + \frac{2}{10^4} + \dots;$$

there is no period in any of these fractions, since otherwise the number would be rational.

The problem of expressing a given function in terms of others, say in terms of the simplest functions, polynomials, leads in a natural way to similar expressions consisting of an infinite set of terms, each being added consecutively to the sum of the previous ones. We shall turn to this problem a little later in art. 3, and meantime deal with the question of series in general.

Suppose we have a sequence $u_1, u_2, \dots, u_n$.

Definition. The expression

$$u_1 + u_2 + \dots + u_n + \dots$$

is called a **series**, $u_1, u_2, \dots, u_n, \dots$ being its **terms**.

If the expression consists of a finite number of terms, the series is said to be finite; whereas if the terms form an infinite sequence, the series is infinite. Of course we are interested here in infinite series; they are usually simply called series, leaving out the word "infinite".

If all the terms of the series are numbers, the series is said to be numerical, whilst it is functional if the terms are functions.

The expression for the n -th term (n arbitrary) is called the general term of the series. A series can be said to be specified if we are given a function of an integral argument n expressing the general term. Thus, if the general term is $u_n = 1/2^n$ or $u_n = 1/n^p$, the initial "segment" of the series is

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8}, \quad \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p},$$

(taking $n = 1, 2, 3$).

II. SUM OF A SERIES. We next define the sum of a series

$$u_1 + u_2 + \dots + u_n + \dots$$

We write s_n for the sum of its first n terms:

$$s_n = u_1 + u_2 + \dots + u_n$$

As n increases indefinitely, an increasingly greater number of terms is included in the sum s_n . This leads naturally to the following definition.

Definition. The **sum** s of a sequence $u_1, u_2, \dots, u_n, \dots$ is the limit (if it exists) of the sum s_n of the first n terms as n tends to infinity:

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} (u_1 + u_2 + \dots + u_n).$$

This is written as:

$$s = u_1 + u_2 + \dots + u_n + \dots$$

and s is called the **sum of the series** $u_1 + u_2 + \dots + u_n, \dots$, s_n being a **partial sum**.

We have already used this definition of sum in the summation of the terms of an infinite geometrical progression (Sec. 28, III) and the terms of the sequence $1, 1/1!, 1/2!, \dots, 1/n!, \dots$ (Sec. 41).

Thus for every series $u_1 + u_2 + \dots + u_n + \dots$ there is a corresponding sequence $s_1, s_2, \dots, s_n, \dots$, where $s_n = u_1 + u_2 + \dots + u_n$; the limit s of this sequence is the "sum" of the series. Conversely, for every convergent sequence $a_1, a_2, \dots, a_n, \dots$, we can form a corresponding series $u_1 + u_2 + \dots + u_n + \dots$, where $u_n = a_n - a_{n-1}$, the sum s of which is the limit of the sequence; in fact,

$$s_n = u_1 + u_2 + \dots + u_n = a_1 + (a_2 - a_1) + \dots + (a_n - a_{n-1}) = a_n$$

and $s = \lim s_n = \lim a_n$. Consequently, finding the sum of a series is simply another form, often very convenient, of passing to the limit in a sequence.

Let the series $u_1 + u_2 + \dots + u_n + \dots$ have a sum s . Then

$$s = s_n + r_n,$$

where $r_n = u_{n+1} + u_{n+2} + \dots$ is an infinite series called the n -th remainder. The remainder of a series is the error that results from taking the n -th partial sum as approximately the sum of the series.

Definition. A series which has a sum, i.e. for which s_n has a limit as $n \rightarrow \infty$, is described as **convergent**. If s_n has no limit (and in particular, tends to ∞), the series is said to be **divergent**.

We can mention as examples of convergent series: the infinite geometrical progression with $|q| < 1$; the series $1 + 1/1! + 1/2! + \dots + 1/n! + \dots$, the sum of which is e .

Examples of divergent series include: the infinite geometrical progression with $|q| > 1$; the series $1 + 3 + 5 + \dots + (2n - 1) + \dots$ (for, $s_n = n^2$, i.e. $s_n \rightarrow \infty$ as $n \rightarrow \infty$); the series $1 - 1 + 1 - 1 + \dots$ (here s_n is equal to 0 or 1, depending on whether n is even or odd, so that s_n has no limit).

III. NECESSARY TEST FOR CONVERGENCE. In certain circumstances convergent series retain the same properties as ordinary finite sums.

Obviously, then, the question of first importance in regard to a series is whether it is convergent or divergent. Our immediate aim is to discover tests so that we can judge from the general term whether a series is convergent or divergent.

REMARK. *When investigating the convergence of a series, we can leave out of account (simply ignore) any finite number of initial terms.* For, let $n = N + m$; let s_N denote the sum of the first N terms, s'_m the sum of the next m terms. Then

$$s_n = s_N + s'_m$$

Taking N as constant, we see that, as $n \rightarrow \infty$, $m \rightarrow \infty$ also, and s_n has a limit or has not, simultaneously with s'_m .

Necessary test for convergence of series. If a series is convergent, the general (n -th) term tends to zero as n increases indefinitely.

Proof. We have:

$$s_n = u_1 + u_2 + \cdots + u_{n-1} + u_n = s_{n-1} + u_n;$$

if the series is convergent, then

$$\lim_{n \rightarrow \infty} s_{n-1} = s \quad \text{and} \quad \lim_{n \rightarrow \infty} s_n = s$$

Hence it follows that

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} (s_n - s_{n-1}) = \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} s_{n-1} = s - s = 0$$

This necessary test for convergence implies the following sufficient test for divergence:

If u_n does not tend to zero, the series cannot be convergent, i.e. it is divergent.

Take say the series

$$\frac{1}{101} + \frac{2}{201} + \frac{3}{301} + \cdots + \frac{n}{100n+1} + \cdots$$

This is divergent, because its general term $u_n = n/(100n+1)$ does not tend to zero as $n \rightarrow \infty$ (it tends to $1/100$).

However, *the fact that the n -th term tends to zero is not a sufficient test for the convergence of a series.*

We take the classical example of the so-called "harmonic series":

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

Although $u_n = 1/n \rightarrow 0$ here, the series is nevertheless divergent. For we can show that $s_n \rightarrow \infty$ as $n \rightarrow \infty$: since $x > \ln(1+x)$ (Sec. 68, III, example 3), we have $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(1+1) + \ln\left(1 + \frac{1}{2}\right) + \cdots + \ln\left(1 + \frac{1}{n}\right)$, i.e.

$$s_n > \ln\left[(1+1)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right)\right] = \ln(n+1);$$

in view of the fact that $\ln(n+1) \rightarrow \infty$ as $n \rightarrow \infty$, s_n also increases indefinitely.

124. Series with positive terms. Sufficient tests for convergence.

We now deal specially with series, all the terms of which are positive. For such series, certain useful sufficient tests for convergence or divergence can be deduced from two simple theorems, relating to the comparison of the given series with another which is known to be convergent or divergent beforehand.

Suppose we have two series with positive terms:

$$\sum_{n=1}^{\infty} u_n, \quad u_n \geq 0, \quad (1)$$

and

$$\sum_{n=1}^{\infty} v_n, \quad v_n \geq 0. \quad (2)$$

THEOREM 1*. If every term of series (1) is not greater than the corresponding term of series (2), i.e. $u_n \leq v_n$ ($n = 1, 2, \dots$) and if series (2) is convergent, (1) is also convergent.

Proof. We put:

$$s_n = \sum_{k=1}^n u_k, \quad \sigma_n = \sum_{k=1}^n v_k;$$

s_n and σ_n do not decrease with increasing index n : s_{n+1} and σ_{n+1} are obtained from s_n and σ_n by the addition of non-negative terms

* In view of the remark on p. 435, theorems 1 and 2 remain valid when the hypotheses are satisfied by terms starting from some $n = N$, rather than for all n .

u_{n+1} and v_{n+1} . Since σ_n tends to a limit by hypothesis — let it be denoted by σ — then $\sigma_n \leq \sigma$ for any n . But $s_n \leq \sigma_n$ also by hypothesis; i.e. $s_n \leq \sigma$ for any n . Thus s_n is a non-decreasing and bounded function as $n \rightarrow \infty$; hence s_n has a limit.

THEOREM 2. If every term of series (1) is not less than the corresponding term of series (2), i.e. $u_n \geq v_n$ ($n = 1, 2, \dots$), and series (2) is divergent, (1) is also divergent.

Proof. Since $\sigma_n = \sum_{k=1}^n v_k$ is not decreasing, the divergence of series (2) can only be due to the fact that $\sigma_n \rightarrow \infty$. But, by hypothesis, $s_n \geq \sigma_n$, i.e. s_n also increases indefinitely as $n \rightarrow \infty$, i.e. series (1) is also divergent. (Theorem 2 also follows easily by reductio ad absurdum from Theorem 1.)

This is what we had to prove.

We take as an example of the application of these theorems the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots,$$

where $p > 0$. We show that this series is divergent if $p \leq 1$ and converges if $p > 1$.

Let $p \leq 1$. Then every term of the series is not less than the corresponding term of the harmonic series:

$$\frac{1}{n^p} \geq \frac{1}{n};$$

since the harmonic series is divergent, our present series must be divergent. For instance, the series $1 + 1/\sqrt{2} + 1/\sqrt{3} + \dots$ is divergent.

Now let $p > 1$. We re-group the terms of the series thus:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \left(\frac{1}{2^p} + \frac{1}{3^p} \right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} \right) + \dots (*)$$

We replace all the terms in each bracket by the greatest (the first); we get

$$1 + \left(\frac{1}{2^p} + \frac{1}{2^p} \right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} \right) + \dots$$

i.e.

$$1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}} \right)^2 + \left(\frac{1}{2^{p-1}} \right)^3 + \dots \quad (**)$$

This series is a geometrical progression with ratio q less than 1 ($q = 1/2^{p-1} < 1$); it is therefore convergent, and since each term of series(*) is less than the corresponding term of series(**), the given series is convergent for $p > 1$.

For instance, the series $1 + 1/2^2 + 1/3^2 + \dots$ is convergent.

By taking a geometrical progression as the comparison series, the last two theorems lead to practically useful sufficient tests for convergence or divergence.

We take the series

$$u_1 + u_2 + \dots + u_n + \dots$$

with positive terms.

I. D'ALEMBERT'S TEST* (GENERAL). (1) If, as from a certain $n = N$, the ratio δ_n of two successive terms:

$$\delta_n = \frac{u_{n+1}}{u_n},$$

is less than q , $q < 1$, i.e. $\delta_n < q$, the series is convergent;

(2) if the ratio δ_n of two successive terms remains not less than 1 as from a certain $n = N$, i.e. $\delta_n \geq 1$, the series is divergent.

Proof. We neglect the first $N - 1$ terms of the series, which we know have no effect on the convergence (Sec. 123).

We now have for $n \geq N$:

In case (1):

$$\begin{aligned} u_{n+1} &< q u_n, \\ u_{n+2} &< q u_{n+1} < q^2 u_n, \end{aligned}$$

and we get the convergent (since $q < 1$) geometrical progression

$$q u_n + q^2 u_n + \dots,$$

each term of which is greater than the corresponding term of the given series. The latter is thus convergent (see Theorem 1).

In case (2):

$$\begin{aligned} u_{n+1} &\geq u_n > 0, \\ u_{n+2} &\geq u_{n+1} \geq u_n, \end{aligned}$$

and we get the divergent series

$$u_n + u_n + \dots,$$

* d'Alembert (1717-1783), a famous French mathematician and encyclopaedist, did a great deal for the development of mathematical analysis. The present refined form of the test is due to Cauchy, not d'Alembert.

each term of which is less than the corresponding term of the given series. The latter is therefore divergent. The divergence is also clear from the fact that $u_{n+m} > u_n > 0$ for any m , so that the necessary convergence test is not satisfied—the general term does not tend to zero.

A particular case of d'Alembert's test is presented by the limiting form of it, which is often very useful in practice. This particular case is also known as d'Alembert's test.

D'ALEMBERT'S TEST (PARTICULAR). If the limit ρ of the ratio u_{n+1}/u_n exists as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \rho,$$

the series is convergent if $\rho < 1$ and divergent if $\rho > 1$.

When $\rho = 1$ the series may be either convergent or divergent.

Proof. Let $\rho < 1$; we can always choose an N such that, with $n > N$, we have

$$\frac{u_{n+1}}{u_n} < \rho + \varepsilon,$$

where $\varepsilon > 0$ is so small that $\rho + \varepsilon$ is still less than 1. But we now conclude by the first part of d'Alembert's general test that the series is convergent.

Let $\rho > 1$; we can choose an N such that, with $n > N$, we have

$$\frac{u_{n+1}}{u_n} > \rho - \varepsilon,$$

where $\varepsilon > 0$ is so small that $\rho - \varepsilon$ is still greater than 1; by the second part of the general test, we conclude that the series is divergent.

Finally, let $\rho = 1$. We take as an example the series $\sum_{n=1}^{\infty} 1/n^p$, for which

$$\delta_n = \frac{n^p}{(n+1)^p} = \left(\frac{n}{1+n} \right)^p \rightarrow 1$$

as $n \rightarrow \infty$, independently of the value of p . At the same time, we know that the series is divergent with $p \leq 1$ and convergent with $p > 1$. Thus we can say nothing about the convergence of the series with $\rho = 1$.

Since the question of a sequence having a limit is equivalent to the question of the corresponding series being convergent or not, d'Alembert's test—and any other convergence or divergence test for series—can be re-stated in reference to sequences. The reader can do this for himself.

II. CAUCHY'S (GENERAL) TEST. (1) If, as from a certain $n = N$, the n -th root α_n of the n -th term of a series:

$$\alpha_n = \sqrt[n]{u_n},$$

is less than q , $q < 1$, i.e. $\alpha_n < q$, the series is convergent;

(2) if the n -th root α_n of the n -th term is not less than 1, i.e. $\alpha_n \geq 1$, as from a certain $n = N$, the series is divergent.

Proof. This proof does not differ in essence from the proof of d'Alembert's test. We have after a certain $n = N$:

In case (1):

$$\begin{aligned} u_n &< q^n, \\ u_{n+1} &< q^{n+1}, \\ &\dots \end{aligned}$$

and we get a convergent (since $q < 1$) geometrical progression

$$q^n + q^{n+1} + \dots,$$

each term of which is greater than the corresponding term of the given series. The latter is therefore convergent (see theorem 1);

In case (2):

$$\begin{aligned} u_n &> 1, \\ u_{n+1} &> 1, \\ &\dots \end{aligned}$$

and the divergence is obvious, since the general term does not tend to zero.

A particular form of Cauchy's test follows in the limit, this being also known as Cauchy's test.

CAUCHY'S (PARTICULAR) TEST. If the limit ρ of $\sqrt[n]{u_n}$ exists as $n \rightarrow \infty$:

$$\lim \sqrt[n]{u_n} = \rho,$$

the series is convergent with $\rho < 1$ and divergent with $\rho > 1$.

When $\rho = 1$ the series may be either convergent or divergent.

The proof is a precise repetition of that for d'Alembert's test, and may be left to the reader.

The tests of d'Alembert and Cauchy are extremely simple and often employed. This is particularly true of d'Alembert's test. At the same time, they are not always usable. We shall give a further test, which is very convenient and sometimes yields an answer when the other two tests do not.

CAUCHY'S INTEGRAL TEST. If the terms of the series

$$u_1 + u_2 + \cdots + u_n + \cdots$$

can be regarded as the values, for integral values of x , of some function $y = f(x)$, which is positive, continuous and decreasing in the interval from $x = 1$ to ∞ :

$$u_1 = f(1), \quad u_2 = f(2), \quad u_3 = f(3), \dots, \quad u_n = f(n),$$

the series is convergent if the improper integral

$$I = \int_1^{\infty} f(x) dx$$

is convergent (exists), and divergent if this integral is divergent (does not exist) (see Secs. 110, 111).

Proof. We take the curvilinear trapezoid bounded by the curve $y = f(x)$, with base from $x = 1$ to $x = n$, where n is a positive integer (Fig. 139). Its area is given by the integral

$$I_n = \int_1^n f(x) dx.$$

We mark off integral points of the base: $x = 1, x = 2, \dots, x = n - 1, x = n$. These points correspond to two step figures (Fig. 139): one of them (the "interior") has an area equal to $f(2) + f(3) + \cdots + f(n) = s_n - u_1$, the other (the "exterior") an area equal to $f(1) + f(2) + \cdots + f(n - 1) = s_n - u_n$. The area of the first figure is less than that of the curvilinear trapezoid, and the area of the second greater, i.e. we have

$$s_n - u_1 < I_n < s_n - u_n.$$

Hence we obtain the two inequalities:

$$s_n < u_1 + I_n, \tag{*}$$

$$s_n > I_n. \tag{**}$$

(1) Let $I = \lim_{n \rightarrow \infty} I_n$ exist; then $I_n < I$, and we find from inequality(*), for any n :

$$s_n < u_1 + I.$$

Thus s_n is an increasing and bounded function, i.e. has a limit, i.e. the series is convergent.

(2) Let I not exist; then $I_n \rightarrow \infty$ as $n \rightarrow \infty$, and we conclude from (**) that s_n also increases indefinitely, i.e. the series diverges.

The example of the already familiar series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

may be used to show that the integral test readily gives an answer when d'Alembert's and Cauchy's tests are useless: both $\delta_n \rightarrow 1$ and $\alpha_n \rightarrow 1$ for this series. Let us apply the integral test.

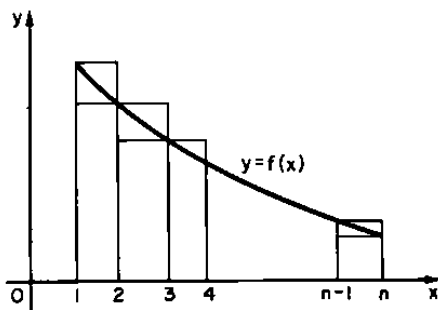


FIG. 139

Obviously, the function $f(x)$ can be taken as $1/x^p$. The integral

$$\int_1^{\infty} \frac{1}{x^p} dx,$$

as we know from Sec. 111, exists or not, depending on whether $p > 1$ or $p \leq 1$. Thus the series is convergent in the first case and divergent in the second.

125. Series with arbitrary terms. Absolute convergence. We turn to series with terms of arbitrary sign. First of all, we consider so-called alternating series. The terms in these are alternately positive and negative. Such a series may be written as

$$\pm (u_1 - u_2 + u_3 - \dots), \quad (1)$$

where $u_1, u_2, u_3, \dots$ are positive numbers. We can obviously take the $+$ sign in front of the series as a whole without loss of generality.

A simple sufficient test for convergence of an alternating series may be mentioned.

LEIBNIZ'S TEST. If the absolute values of the terms decrease in an alternating series, i.e. $u_1 > u_2 > u_3 > \dots$ in series (1), and the general term u_n tends to zero, the series is convergent, and the absolute value of its sum does not exceed the first term; the remainder r_n has an absolute value not exceeding the first of the neglected terms.

Proof. Let n run through the even numbers: $n = 2m$. Then

$$s_{2m} = (u_1 - u_2) + (u_3 - u_4) + \dots + (u_{2m-1} - u_{2m}).$$

By hypothesis, the expression in each bracket is positive, i.e. s_{2m} is positive, and increases with increasing m . On the other hand:

$$s_{2m} = u_1 - [(u_2 - u_3) + (u_4 - u_5) + \dots + u_{2m}].$$

Again by hypothesis, the sum in the square bracket is positive; hence $s_{2m} < u_1$ for any m . Thus s_{2m} is increasing and bounded, i.e. there exists $\lim_{m \rightarrow \infty} s_{2m} = s$, where $s < u_1$.

Now let $n = 2m + 1$. We have:

$$s_{2m+1} = s_{2m} + u_{2m+1}.$$

As $m \rightarrow \infty$, $u_{2m+1} \rightarrow 0$ by hypothesis, so that

$$\lim_{m \rightarrow \infty} s_{2m+1} = \lim_{m \rightarrow \infty} s_{2m} = s.$$

Thus s_n has a limit s less than u_1 as $n \rightarrow \infty$. The remainder

$$r_n = \pm (u_{n+1} - u_{n+2} + \dots)$$

is a series satisfying all the conditions of Leibniz's test. Its sum r_n is therefore less, in absolute value, than the first term in the bracket, i.e. $|r_n| < u_{n+1}$.

This is what we had to prove.

When applicable, Leibniz's test enables us not merely to establish the convergence of the series but also to estimate the error in neglecting all the terms after a certain number.

For instance, it is immediately clear from Leibniz's test that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

is a convergent series (we shall see later, in Sec. 130, that its sum is $\ln 2$), the error of the approximation $\ln 2 \approx 1 - \frac{1}{2} + \frac{1}{3} - \dots \pm 1/(n-1)$ being $1/n$.

We shall mention one important general test for series in which the signs of the terms are distributed arbitrarily.

SUFFICIENT TEST FOR CONVERGENCE. If the series consisting of the absolute values of the terms of a given series is convergent, the latter is also convergent.

Proof. Let s_n denote the sum of the first n terms of the series

$$u_1 + u_2 + \dots + u_n + \dots$$

let s_n^+ be the sum of all the positive, and s_n^- the sum of the absolute values of all the negative terms among the first n in the series. Then

$$s_n = s_n^+ - s_n^- \quad \text{and} \quad \sigma_n = s_n^+ + s_n^-,$$

where

$$\sigma_n = |u_1| + |u_2| + \dots + |u_n|.$$

Since σ_n has a limit (say σ) by hypothesis, whilst s_n^+ and s_n^- are positive and increasing functions of n , and $s_n^+ < \sigma_n < \sigma$ and $s_n^- < \sigma_n < \sigma$, they also have a limit. In view of this, $s_n = s_n^+ - s_n^-$ also has a limit as $n \rightarrow \infty$.

This is what we had to prove.

This sufficient test is not necessary, i.e. the series $\sum_{k=1}^{\infty} u_k$ may be convergent even though $\sum_{k=1}^{\infty} |u_k|$ is divergent. For instance,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

is a convergent series, whilst the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots,$$

consisting of the absolute values of its terms, is known to be divergent.

Definition. A series such that the absolute values of its terms form a convergent series is said to be *absolutely* or *unconditionally convergent*.

If the series is convergent, but the series formed from the absolute values of its terms is divergent, the given series is said to be *non-absolutely* or *conditionally convergent*.

For instance, the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is conditionally convergent.

The tests of Sec. 124 may be applied, as for series with positive terms, to the series formed by the absolute values of the terms of a

given series. If the test reveals convergence, this implies absolute convergence of the given series. Whereas if d'Alembert's and Cauchy's tests reveal divergence of the series of absolute values, the given series must also be divergent. For d'Alembert's and Cauchy's tests are based on the fact that the general term does not tend to zero, and clearly, if $\lim |u_n| \neq 0$, then $\lim u_n \neq 0$ also, i.e. $\sum u_n$ is divergent. Thus to establish the convergence or divergence of the given series (with terms of arbitrary sign), we can apply the tests of d'Alembert and Cauchy to the series consisting of the absolute values of the terms of the given series.

The distinction noticed between absolute and non-absolute convergence is extremely important. Certain important properties of finite sums can be carried over only to absolutely convergent series; conditionally convergent series do not possess some of these properties.

126. Operations on series. We shall now give some extremely simple rules for operations on convergent series, and the elementary properties of these.

I. When a convergent series

$$s = u_1 + u_2 + \dots + u_n + \dots$$

is multiplied by a number λ , this means that each term of the series is multiplied by λ , giving the new series

$$\lambda u_1 + \lambda u_2 + \dots + \lambda u_n + \dots$$

This series is also convergent, its sum being λs .

We mean by the sum of two convergent series

$$s' = u_1 + u_2 + \dots + u_n + \dots$$

and

$$s'' = v_1 + v_2 + \dots + v_n + \dots$$

the series formed by adding corresponding terms of the given series:

$$(u_1 + v_1) + (u_2 + v_2) + \dots + (u_n + v_n) + \dots$$

This series is also convergent, its sum being $s' + s''$.

Multiplication of two series will be considered later in this section.

II. *The terms of a convergent series can be grouped in any manner, i.e. the series resulting from the grouping is also convergent and has the same sum.*

Let

$$s = u_1 + u_2 + \dots + u_n + \dots$$

be a convergent series. We form a new series by grouping successive terms as follows: $(u_1 + u_2 + \dots + u_{n_1}) + (u_{n_1+1} + \dots + u_{n_2}) + \dots + (u_{n_m} + \dots + u_{n_k}) + \dots$; this series can be written as

$$v_1 + v_2 + \dots + v_k + \dots$$

where $v_1, v_2, \dots, v_k, \dots$ denote the values of the 1st, 2nd, $\dots$, k -th $\dots$ brackets. This series is likewise convergent, its sum being s . (We omit the proof.)

Thus every convergent series possesses the associative property.

III. The commutative property is a more difficult matter.

The terms of an absolutely convergent series can be moved about in any manner, i.e. the commutation results in a series which is also absolutely convergent, to the same sum.

It will be noticed that the commutative property is asserted only with a special condition: the series must be absolutely convergent.

Let

$$s = u_1 + u_2 + \dots + u_n + \dots \quad (*)$$

be an absolutely convergent series. We form a new series by shifting an indefinite sequence of terms:

$$u_{n_1} + u_{n_2} + \dots + u_{n_k} + \dots, \quad (**)$$

where $n_1, n_2, \dots, n_k, \dots$ are positive integers taken in any order. This latter series is also absolutely convergent, and its sum is s . (We again omit the proof.)

Thus the sum of an absolutely convergent series is unchanged on commuting its terms.

This property, which is so obvious for finite sums, does not in fact hold for non-absolutely convergent series. Let us take say the non-absolutely convergent series (see Sec. 125):

$$A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots$$

and multiply it by $\frac{1}{2}$:

$$\frac{1}{2}A = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \dots$$

We obtain on adding the two series:

$$\frac{3}{2}A = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots$$

This series can be got from the original one simply by re-arranging the terms. A change of the order in which the terms follow each other in a conditionally convergent series thus leads to a change in the sum. We can expect some other re-arrangement of the terms to lead to another change in the sum. The following theorem of Riemann may be mentioned apropos of this:

It is always possible to re-arrange the terms of a conditionally convergent series in such a way that the series converges to any previously assigned number or even becomes divergent.

IV. We shall understand by the product of two convergent series

$$s' = u_1 + u_2 + \dots + u_n + \dots, \quad (1)$$

$$s'' = v_1 + v_2 + \dots + v_n + \dots \quad (2)$$

the series formed from all the possible pairs of products of terms of the series, arranged in the following order:

$$\begin{aligned} & (u_1 v_1) + (u_1 v_2 + u_2 v_1) + (u_1 v_3 + u_2 v_2 + u_3 v_1) + \dots \\ & \dots + (u_1 v_n + u_2 v_{n-1} + \dots + u_{n-1} v_2 + u_n v_1) + \dots \end{aligned} \quad (3)$$

In each bracketed group of terms the sum of the subscripts of the factors (i.e. the numbers which indicate their positions in the original series) is constant: 2 in the first bracket, 3 in the second, 4 in the 3rd, . . . , $n + 1$ in the n -th.

The theorem holds:

If series (1) and (2) are absolutely convergent, their product is also absolutely convergent, its sum s being equal to the product of the sums of (1) and (2):

$$s = s' \cdot s''.$$

A proof may be found in more comprehensive works on analysis.

Thus the properties of finite series and the arithmetical operations cannot be extended wholesale to infinite series. Generally speaking, a subsidiary condition is required, such as the absolute convergence of the infinite series.

2. Functional Series

127. Definitions. Uniform convergence. Now let

$$u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots \quad (\text{A})$$

be a given functional series; its terms are functions of an independent variable and are defined in a certain domain.

Series (A) may be convergent for some values of x and divergent for others. A value $x = x_0$ for which the numerical series

$$u_1(x_0) + u_2(x_0) + \cdots + u_n(x_0) + \cdots$$

is convergent, is called a point of convergence of series (A). The set of points of convergence of the series is called its domain of convergence; the series is said to be convergent in this domain. The domain of convergence is generally some interval of Ox .

The sum of the series is a function of x , defined in the domain of convergence; we write it as $f(x)$:

$$f(x) = u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots.$$

The series is said to be convergent to the function $f(x)^*$.

The question naturally arises of establishing the properties of the function $f(x)$ represented by the series from the known properties of the terms—the functions $u_1, u_2, \dots, u_n, \dots$.

We remark first of all that the continuity of $u_1, u_2, \dots, u_n, \dots$ is no longer sufficient for asserting the continuity of $f(x)$. We shall give below an example of a series of continuous functions which converges to a discontinuous function. The convergence of a series of continuous functions to a continuous function is ensured by a supple-

* The series is also said to define, or express, or represent $f(x)$.

mentary condition, expressing an important feature of the convergence of functional series. The feature is as follows. Let x_1 be a point inside the domain of convergence. We write:

$$f(x_1) = s_n(x_1) + r_n(x_1),$$

where $s_n(x_1)$ is the sum of the first n terms and $r_n(x_1)$ the remainder of the series. Since $\lim_{n \rightarrow \infty} s_n(x_1) = f(x_1)$, i.e. $\lim_{n \rightarrow \infty} r_n(x_1) = 0$, given any $\varepsilon > 0$, a positive N_1 can be chosen so that

$$|r_n(x)| < \varepsilon \quad (*)$$

for $x = x_1$ and $n \geq N_1$.

We now take another point x_2 of the domain of convergence. Inequality $(*)$ will generally speaking be fulfilled for x_2 with the same ε as from some other $n = N_2$. Obviously, if n exceeds the greater of N_1 and N_2 , $(*)$ will be satisfied for x_2 as well as x_1 . We can evidently draw the same conclusion for any finite number of fixed points of the domain of convergence. But if we are concerned with all the points of the domain (i.e. any of them), we cannot assert in advance the existence of an N such that $(*)$ is satisfied for all $n \geq N$, no matter what the point x . For we have as yet no basis for excluding the possibility that the necessary $n = N_i$ increases on passing from one x_i to another; and if there is an infinite set of x_i , and the N_i increase indefinitely, it may happen that no common value $n = N$ can be chosen for all the x_i , as from which inequality $(*)$ is fulfilled.

Definition. A convergent functional series $u_1(x) + u_2(x) + \dots + u_n(x) + \dots$ is said to be uniformly convergent in a domain if, given any arbitrarily small $\varepsilon > 0$, there is a corresponding positive integer N such that the n -th remainder $r_n(x) = u_{n-1}(x) + u_{n-2}(x) + \dots$ remains less in absolute value than ε , $|r_n(x)| < \varepsilon$, for $n > N$, whatever the x in the domain.

This property points to the so-to-speak homogeneous nature of the convergence of the series at all points of the domain, this being emphasized by the term "uniform"*. If the series $f(x) = u_1(x) + u_2(x) + \dots + u_n(x) + \dots$ is uniformly convergent in an inter-

* "Uniform" continuity (see Sec. 37) also defines the "homogeneous" nature of the continuity of a function at every point of the corresponding interval. The "uniformity" of any property plays an important role in mathematics. Speaking very generally, we can say that this is explained by the possibility of using the property, when uniform, throughout a domain (integrally) and not merely at a single point (locally).

val, $f(x)$ —the sum of the series—can be approximated with the aid of a partial sum:

$$s_N(x) = u_1(x) + u_2(x) + \cdots + u_N(x)$$

with the same accuracy at all points of the interval in question; this accuracy is characterized by the inequality $|r_N(x)| < \varepsilon$, valid for any x , N being chosen in accordance with the previously assigned ε .

It is obvious that the concept of uniform convergence can be carried over to functional sequences: *a sequence of functions $a_1(x)$, $a_2(x)$, $\dots$, $a_n(x)$, $\dots$ is uniformly convergent to a function $f(x)$ in a given domain if, given any $\varepsilon > 0$, an N can be chosen such that $|f(x) - a_n(x)| < \varepsilon$ throughout the domain for $n \geq N$.*

The fact that the $a_n(x)$ tend uniformly to their limit function $f(x)$ is interpreted geometrically as follows: if we draw parallel curves above and below the curve $y = f(x)$ separated from it by an arbitrarily small distance ε ($\varepsilon > 0$) measured along Oy (i.e. the curves $y = f(x) \pm \varepsilon$), as from a certain $n = N$, depending on the ε chosen, the curves $y = a_n(x)$ will lie wholly inside the resulting strip of width 2ε (Fig. 140).

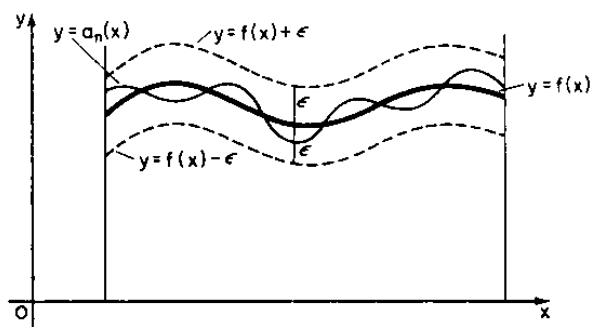


FIG. 140

The following general theorem brings out the value of the concept of uniform convergence.

THEOREM. *The sum of a series which is uniformly convergent in a domain and consists of continuous functions is itself a continuous function in this domain.*

Proof. Let the series $u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots$ be uniformly convergent in some domain. We write

$$f(x) = s_n(x) + r_n(x)$$

and take an arbitrarily small $\varepsilon > 0$. By virtue of the uniform convergence, an N can be found such that, with $n \geq N$, $|r_n(x)| < \varepsilon/3$ for any x of the domain. Let x be given the increment h ; we consider the difference

$$f(x+h) - f(x) = [s_n(x+h) - s_n(x)] + r_n(x+h) - r_n(x).$$

We have:

$$|f(x+h) - f(x)| \leq |s_n(x+h) - s_n(x)| + |r_n(x+h)| + |r_n(x)|.$$

By what has been said, with $n \geq N$ we have

$$|r_n(x+h)| < \frac{\varepsilon}{3}, \quad |r_n(x)| < \frac{\varepsilon}{3}.$$

But $s_n(x)$, being the sum of a finite number of continuous functions for any assigned and fixed n , is a continuous function. Thus, for sufficiently small δ ,

$$|s_n(x+h) - s_n(x)| < \frac{\varepsilon}{3},$$

provided $|h| < \delta$. Hence, with $|h| < \delta$,

$$|f(x+h) - f(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This proves the continuity of the function $f(x)$.

We shall give an example showing that the uniform convergence condition is essential for the truth of the theorem.

The series

$$\frac{x}{1+x} + \frac{x}{(1+x)^2} + \cdots + \frac{x}{(1+x)^n} + \cdots$$

consists of functions continuous in any closed interval $[0, a]$, $a > 0$, and is convergent for all non-negative x . For, when $x > 0$, it is a geometrical progression with ratio $q = \frac{1}{1+x} (> 1)$. The sum of the series, with $x > 0$, is

$$f(x) = \frac{\frac{x}{1+x}}{1 - \frac{1}{1+x}} = 1.$$

As the same time, $f(0) = 0$, since each term vanishes with $x = 0$. We see that the sum $f(x)$ of the series is a discontinuous function (at $x = 0$): $f(x) = 1$ if $x > 0$, and $f(0) = 0$. And in fact, the series is non-uniformly convergent in

the interval $[0, a]$. For, whatever the ε , $0 < \varepsilon < 1$, and however large the N , a value of x sufficiently close to zero can always be found such that

$$r_N(x) = \frac{x}{(1+x)^{N+1}} + \frac{x}{(1+x)^{N+2}} + \cdots = \frac{1}{(1+x)^N}$$

will be greater than ε . For this, we have to take $x < (1/\varepsilon)^{1/N} - 1$, as may easily be verified.

A simple sufficient test for the uniform convergence of functional series may be given.

WEIERSTRASS'S TEST*. If functions $u_1(x), u_2(x), \dots, u_n(x), \dots$ respectively do not exceed in absolute value the positive numbers $M_1, M_2, \dots, M_n, \dots$ in some domain, and if the numerical series

$$M_1 + M_2 + \cdots + M_n + \cdots$$

is convergent, the functional series $u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots$ is uniformly (and absolutely) convergent in this domain.

Proof. We observe first of all that the series

$$u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots$$

is absolutely convergent, since the absolute values of all its terms do not exceed the corresponding terms of the convergent series

$$\sum_{k=1}^{\infty} M_k \text{ (see Sec. 125).}$$

Further, given $\varepsilon > 0$, an N can be chosen such that, with $n \geq N$:

$$M_{n+1} + M_{n+2} + \cdots < \varepsilon.$$

But the remainder of the series

$$r_n(x) = u_{n+1}(x) + u_{n+2}(x) + \cdots$$

has an absolute value which, by hypothesis, satisfies at any point of the domain the inequality

$$\begin{aligned} |r_n(x)| &= |u_{n+1}(x) + u_{n+2}(x) + \cdots| \leq |u_{n+1}(x)| + |u_{n+2}(x)| + \cdots \\ &\leq M_{n+1} + M_{n+2} + \cdots. \end{aligned}$$

Hence, provided $n \geq N$, we have

$$|r_n(x)| < \varepsilon,$$

which holds whatever the x in the domain. It follows from this that the series is both absolutely and uniformly convergent.

* K. Weierstrass (1815–1897) was a celebrated German mathematician.

Certain authors call functional series satisfying the conditions of Weierstrass's test regularly convergent.

123. Integration and differentiation of functional series. We turn next to applying the basic operations of analysis to convergent functional series.

We prove two general theorems.

THEOREM 1. Let the series $u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots$, consisting of continuous functions, be uniformly convergent in a domain to the function $f(x)$; then the series consisting of the integrals of the original terms over the interval $[a, x]$ belonging to the domain:

$$\int_a^x u_1(x) dx + \int_a^x u_2(x) dx + \cdots + \int_a^x u_n(x) dx + \cdots,$$

will be uniformly convergent to the integral of the sum $f(x)$.

More briefly:

A uniformly convergent series of continuous functions can be integrated term by term.

Proof. Let

$$f(x) = s_n(x) + r_n(x).$$

Then

$$\int_a^x f(x) dx = \int_a^x s_n(x) dx + \int_a^x r_n(x) dx$$

or

$$\int_a^x f(x) dx = \sigma_n(x) + \varrho_n(x)$$

where

$$\sigma_n(x) = \int_a^x s_n(x) dx = \int_a^x u_1(x) dx + \int_a^x u_2(x) dx + \cdots + \int_a^x u_n(x) dx,$$

and

$$\varrho_n(x) = \int_a^x r_n(x) dx.$$

We take $\varepsilon > 0$. By the uniform convergence, an N can be chosen such that, with $n \geq N$,

$$|r_n(x)| < \frac{\varepsilon}{|b-a|}$$

independently of the choice of x . We now have, by the theorem on estimating the absolute value of an integral (Sec. 90):

$$|\varrho_n(x)| = \left| \int_a^x r_n(x) dx \right| \leq |x - a| \frac{\varepsilon}{|b - a|} \leq |b - a| \frac{\varepsilon}{|b - a|} = \varepsilon.$$

Thus, given any $\varepsilon > 0$, we can choose an N such that, with $n \geq N$,

$$|\varrho_n(x)| < \varepsilon,$$

this inequality being uniform with respect to x , i.e. we have uniformly

$$\sigma_n(x) \rightarrow \int_a^x f(x) dx.$$

This is what we wished to prove.

The uniform convergence of the series is an essential factor in the proof of this theorem, which may not hold without it.

The result obtained can also be written as

$$\int_a^x f(x) dx = \lim_{n \rightarrow \infty} \int_a^x s_n(x) dx,$$

or, on taking into account the fact that $f(x) = \lim_{n \rightarrow \infty} s_n(x)$, as

$$\lim_{n \rightarrow \infty} \int_a^x s_n(x) dx = \int_a^x \lim_{n \rightarrow \infty} s_n(x) dx,$$

i.e. the operation of uniform passage to the limit and the operation of integration can be commuted.

THEOREM 2. Let the series $u_1(x) + u_2(x) + \dots + u_n(x) + \dots$ consist of functions possessing continuous derivatives; if the series consisting of the derivatives of the terms:

$$u'_1(x) + u'_2(x) + \dots + u'_n(x) + \dots$$

is uniformly convergent in a domain, its sum is the derivative of the sum of the original series in the domain.

More briefly,

A series can be differentiated term by term if the series of derivatives is uniformly convergent.

Proof. Let $F(x)$ be the sum of the "derived" series (which is uniformly convergent by hypothesis):

$$f(x) = u'_1(x) + u'_2(x) + \dots + u'_n(x) + \dots \quad (\dagger)$$

Obviously, we only have to show that $F(x) = f'(x)$ if

$$f(x) = u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots$$

We apply Theorem 1 to series($\dagger$), for which the conditions of the theorem are fulfilled; we get a uniformly convergent series:

$$\begin{aligned} \int_a^x F(x) dx &= \int_a^x u'_1(x) dx + \int_a^x u'_2(x) dx + \cdots + \int_a^x u'_n(x) dx + \cdots \\ &= [u_1(x) - u_1(a)] + [u_2(x) - u_2(a)] + \cdots + [u_n(x) - u_n(a)] + \cdots \end{aligned}$$

Thus

$$\begin{aligned} \int_a^x F(x) dx &= [u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots] - \\ &\quad - [u_1(a) + u_2(a) + \cdots + u_n(a) + \cdots] = f(x) - f(a) \end{aligned}$$

i.e.

$$F(x) = f'(x).$$

This is what we wanted to prove.

The course of the proof shows that the hypotheses imply, by Theorem 1, the uniform convergence of the given series.

It will be observed that the theorem on differentiation of an infinite series is more complicated than the integration theorem—it requires a verification of the fact that the series resulting from differentiation, i.e. the series of derivatives, is uniformly convergent.

The fact that the latter condition cannot be dispensed with is revealed by examples. Thus the series

$$\frac{\sin x}{1} + \frac{\sin 2^3 x}{2^2} + \cdots + \frac{\sin n^3 x}{n^2} + \cdots$$

is uniformly convergent in any interval, because the absolute values of its terms are not greater than the corresponding terms of the convergent series

$$1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \cdots$$

(see theorem 2 in Sec. 127). At the same time, the “derived” series

$$\cos x + 2 \cos 2^3 x + \cdots + n \cos n^3 x \dots$$

is divergent for any x (because the general term does not tend to zero).

We can thus conclude our study of the general theory of series by asserting that *the applicability of the operations of analysis to infinite functional series is guaranteed by the property of uniform convergence of the series.*

It may be remarked, in addition, that *the applicability of the arithmetic operations to infinite series is guaranteed by the property of absolute convergence of the series.*

3. Power Series

129. Taylor's series.

Definition. A **power series** is a functional series of the form

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots,$$

each term being the product of a constant and a power (with positive integral exponent) of the difference $x - x_0$ (in particular, if $x_0 = 0$ — of the independent variable x itself).

The constants $a_0, a_1, a_2, \dots, a_n, \dots$ are called the *coefficients of the power series*.

We naturally arrive at power series when we use Taylor's formula to form approximations to functions by polynomials (Sec. 79) with indefinitely increasing accuracy (i.e. with $\delta_n \rightarrow 0$).

If $f(x)$ has derivatives up to and including order $(n + 1)$ in a neighbourhood of x_0 , as we know,

$$\begin{aligned} f(x) = & f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \cdots \\ & \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n + \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x - x_0)^{n+1}, \end{aligned}$$

where ξ is an intermediate point between x_0 and x , or

$$f(x) = N_n(x - x_0) + R_n \quad (*)$$

(see Sec. 78). For a given n , the approximate expression for $f(x)$ as a polynomial $N_n(x - x_0)$:

$$f(x) \approx N_n(x - x_0)$$

is generally speaking the more accurate, the smaller the length of the interval in which the approximation is considered (since the error δ_n will be less). Now let the interval $[a, b]$, in which the approximation is considered, remain unchanged. Then an increase in accuracy can often be achieved by increasing the order n of the Taylor formula. For, it is clear from the expression for the maximum error (see Sec. 79):

$$\delta_n = \frac{M_{n+1}}{(n+1)!} (b - a)^{n+1}$$

where $M_{n+1} \geq |f^{(n+1)}(x)|$, that the denominator increases rapidly with increase of n , and δ_n can be expected to tend to zero.

Thus we let n increase indefinitely; we now have to assume, for the validity of Taylor's formula, that derivatives of any order of $f(x)$ exist in the interval $[a, b]$; in addition, we assume that $\lim_{n \rightarrow \infty} \delta_n = 0$, i.e. that $\lim_{n \rightarrow \infty} R_n = 0$ for any x , $a \leq x \leq b$. We obtain from equation (*):

$$f(x) = \lim_{n \rightarrow \infty} N_n(x - x_0) = \lim_{n \rightarrow \infty} \left[f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n \right];$$

in accordance with the definition of the sum of a series, $f(x)$ is the sum of an infinite power series:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \cdots \\ \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n + \cdots \quad (\dagger)$$

The series on the right-hand side is generally called the Taylor series of the function $f(x)$.

Definition. The **Taylor series** of a function $f(x)$ in a neighbourhood of the point x_0 is the power series in the difference $x - x_0$, the coefficients $a_0, a_1, a_2, \dots, a_n, \dots$ of which are given in terms of the derivatives of $f(x)$ at x_0 by

$$a_0 = f(x_0), a_1 = f'(x_0), a_2 = \frac{1}{2!} f''(x_0), \dots, a_n = \frac{1}{n!} f^{(n)}(x_0), \dots$$

These are called the Taylor coefficients of $f(x)$ at x_0 .

If the series on the right-hand side of ($\dagger$) is convergent, this equation can be regarded as so-to-speak the Taylor formula of "infinite order", giving the representation of the function as a polynomial of "infinite degree". The significance of this remarkable representation consists in its combining the simplicity of the (power!) functions from which the Taylor series is formed with the simplicity of the (ordinary!) rules by which the various operations on them are carried out. Thanks to this, the investigation of even very complicated functions (as regards their analytic structure) and the operations on them can be reduced to the elementary "algebra" of "finite" and "infinite"

polynomials. We shall encounter examples of this reduction in the present article and later chapters.

Definition. A function which can be represented in an interval by its convergent Taylor series is called *analytic* in the interval.

As we have seen, the conditions in which a function is analytic are as follows: (1) *it must be "infinitely differentiable" in the interval* (i.e. have derivatives of all orders), (2) *the remainder term of its Taylor formula must tend to zero at any point of the interval on indefinite increase of the order of the formula.*

The first condition enables the Taylor series to be formed, the second guarantees the convergence of the series to the original function.

It may be remarked that every elementary function is analytic throughout its domain of definition, with the possible exception of individual points.

130. Examples. We shall consider some important examples (see Sec. 79) of the expansion of functions in Taylor series.

(1) We have:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + R_n,$$

where $R_n = e^{\xi}x^{n+1}/(n+1)!$, $\xi = \theta x$, $0 < \theta < 1$. Thus

$$|R_n| \leq \frac{1}{(n+1)!} e^M M^{n+1} = \delta_n, \quad \text{where } |x| \leq M.$$

The fact that $\delta_n \rightarrow 0$ as $n \rightarrow \infty$ is shown most simply thus: the numerical series with general term $M^n/n!$ (M is any number) is convergent, as may be verified by d'Alembert's test: $\frac{M^{n+1}}{(n+1)!} \cdot \frac{n!}{M^n} = \frac{M}{n+1} \rightarrow 0$; thus the general term $M^n/n!$ of the series must tend to zero.

Thus, in view of the fact that M is arbitrary, the series

$$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots$$

is convergent for any x , i.e. throughout Ox . This series is of great importance and is sometimes called the exponential series.

In particular, when $x = 1$ we find a series for the number e :

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots,$$

which we have already obtained in Sec. 41.

(2) We have:

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + \frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1} + R_n,$$

where $R_n = \frac{1}{(2n)!} \sin(\xi + n\pi) x^{2n}$, $\xi = \theta x$, $0 < \theta < 1$. Thus

$$|R_n| \leq \frac{1}{(2n)!} M^{2n} = \delta_n, \quad \text{where } |x| \leq M;$$

in view of the fact that $\delta_n \rightarrow 0$ as $n \rightarrow \infty$, as proved in (1), we find for the Taylor series of $\sin x$ in the neighbourhood of $x = 0$:

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + \frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1} + \cdots;$$

the equation holds throughout Ox .

Similarly, we find that

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots,$$

throughout Ox .

These representations of $\sin x$ and $\cos x$ as "infinite" polynomials clearly reveal the oddness of the first function and the evenness of the second.

(3) We have for the function $\ln(1+x)$:

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + \frac{(-1)^{n-1}}{n}x^n + R_n,$$

where $R_n = \frac{(-1)^n}{n+1} \cdot \frac{1}{(1+\xi)^{n+1}} x^{n+1}$, $\xi = \theta x$, $0 < \theta < 1$. Thus

$$|R_n| < \frac{1}{n+1} = \delta_n,$$

if $0 \leq x \leq 1$. It is immediately clear that $\delta_n \rightarrow 0$ as $n \rightarrow \infty$, and we get

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + \frac{(-1)^{n-1}}{n}x^n + \cdots.$$

This series is convergent to $\ln(1+x)$ not only in the interval $[0, 1]$ but also in $(-1, 0)$, as will be shown below with the aid of another approach to the problem of expanding functions as Taylor series.

In particular, we obtain with $x = 1$ the so-called Leibniz series (see Sec. 125):

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n} + \cdots$$

With $x = -1$ the series is divergent, because it represents the familiar harmonic series, with the signs of all the terms changed from $+$ to $-$. Obviously we cannot speak of the series with $x \leq -1$; it is divergent for $x > 1$ (see Sec. 131).

(4) Let us form the Taylor series for the binomial $(1+x)^m$, where m is any number. We have

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \cdots + \frac{m(m-1) \cdots (m-n+1)}{n!} x^n + R_n,$$

$$\text{where } R_n = \frac{m(m-1) \cdots (m-n)}{(n+1)!} (1+\xi)^{m-n-1} x^{n+1}, \xi = \theta x, 0 < \theta < 1;$$

bearing in mind the indefinite increase of n , we can assume that, if $0 \leq x < 1$, then

$$|R_n| \leq \frac{|m(m-1) \cdots (m-n)|}{(n+1)!} x^{n+1} = \delta_n.$$

For, $(1+\xi)^{m-n-1} < 1$ with sufficiently large n .

We show, as in example (1), that $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. In fact, the series with general term δ_n is convergent, because, by d'Alembert's test,

$$\frac{|m-n-1|}{n+1} x \rightarrow x < 1;$$

hence the general term of the series δ_n must tend to zero, i.e. we arrive at the expansion for the interval $[0, 1]$:

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \cdots + \frac{m(m-1) \cdots (m-n+1)}{n!} x^n + \cdots$$

This is the so-called binomial series. It converges to the function $(1+x)^m$ not only in the interval $[0, 1]$, but also in the interval

$(-1, 0)$, as we shall prove a little later. We remark that the binomial series is convergent for any $m > 0$ to the binomial represented by it in the closed interval $[-1, 1]$.

Let us write down the commonly encountered binomial series corresponding to $m = -1, \frac{1}{2}, -\frac{1}{2}$ (the intervals in which the expansions hold are indicated on the right):

$$\begin{aligned}\frac{1}{1+x} &= 1 - x + x^2 - x^3 + \dots & (-1 < x < 1); \\ \sqrt{1+x} &= 1 + \frac{1}{2}x - \frac{1}{2 \cdot 4}x^2 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^4 + \dots + \\ &\quad + (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \dots (2n-3)}{2 \cdot 4 \cdot 6 \cdot 8 \dots 2n} x^n + \dots & (-1 \leq x \leq 1); \\ \frac{1}{\sqrt{1+x}} &= 1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}x^4 - \dots + \\ &\quad + (-1)^n \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \cdot 8 \dots 2n} x^n + \dots & (-1 < x \leq 1).\end{aligned}$$

For $x > 0$, the error due to neglecting all the terms as from the n -th in these three series can be estimated from the absolute value of the n -th term, in accordance with Leibniz's test.

We use another method in the next section to consider the Taylor expansions of some other elementary functions.

131. Interval and radius of convergence.

I. As a result of the various operations on Taylor series, power series may originate which represent functions not previously known to us. It is therefore important to know how to investigate power series *per se*.

Suppose we have the power series

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \quad (\dagger)$$

(We can take such a series so as to simplify the exposition, because any power series $a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$ can be transformed into this form by substituting $x - x_0 = x'$.)

We first investigate the form of the domain of convergence of a power series. It is immediately obvious that three cases are possible:

(1) the domain of convergence consists of a single point $x = 0$ (at which the series reduces to its first term); in other words, the series is divergent for all values of x (except one);

(2) the domain of convergence consists of all points of Ox ; in other words, the series is convergent for all x ;

(3) the domain of convergence consists of more than one point of Ox , whilst there are points of Ox which do not belong to the domain.

The first case can be illustrated by the series

$$1 + x + 2^2x^2 + 3^3x^3 + \cdots + n^n x^n + \cdots;$$

in fact, if $x \neq 0$, as from a sufficiently large n we have $|nx| > 1$, i.e. $|n^n x^n| > 1$, which implies that the general term does not tend to zero; we can take as an example of the 2nd case the exponential series (see Sec. 130, example 1):

$$1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots;$$

the geometric progression

$$1 + x + x^2 + \cdots + x^n + \cdots$$

is an example of a series of case (3).

As a matter of fact—and this is extremely important—the domain of convergence of a power series in the 3rd case is simply an interval of Ox symmetrical about $x = 0$ (symmetrical about $x = x_0$ for the series $a_0 + a_1(x - x_0) + \cdots$). We shall prove this below.

We first consider the following theorem.

ABEL'S THEOREM*. If power series (†) is convergent for $x = x_0 \neq 0$, it is (absolutely) convergent for every x with an absolute value less than x_0 , $|x| < |x_0|$, i.e. in the interval $(-|x_0|, |x_0|)$.

Proof. We notice that, due to the convergence of $\sum_{k=0}^{\infty} a_k x_0^k$, its general term must tend to zero; therefore all the terms of this series are uniformly bounded: a positive constant M exists such that, for any n ,

$$|a_n x_0^n| < M^{**}.$$

* N. Abel (1802–1829) was a famous Norwegian mathematician who accomplished a great deal in his short life for the development of various branches of mathematics.

** If $a_n x_0^n$ tends to zero, as from a certain $n = N$ all the $|a_n x_0^n|$ do not exceed a previously assigned M_1 ; as regards the first N terms: $|a_0|, |a_1 x_0|, |a_2 x_0^2|, \dots, |a_{n-1} x_0^{n-1}|$, let M_2 be a number greater than all these terms. We now have, in fact, $|a_n x_0^n| < M$, where M is the greater of M_1, M_2 , for any n .

We write series(†) as

$$a_0 + a_1 x_0 \left(\frac{x}{x_0} \right) + a_2 x_0^2 \left(\frac{x}{x_0} \right)^2 + \cdots + a_n x_0^n \left(\frac{x}{x_0} \right)^n + \cdots,$$

and take the series formed from the absolute values of its terms:

$$|a_0| + |a_1 x_0| \left| \frac{x}{x_0} \right| + |a_2 x_0^2| \left| \frac{x}{x_0} \right|^2 + \cdots + |a_n x_0^n| \left| \frac{x}{x_0} \right|^n + \cdots$$

By the remark just made, every term is here less than the corresponding term of the geometrical progression with ratio $|x/x_0|$:

$$M + M \left| \frac{x}{x_0} \right| + M \left| \frac{x}{x_0} \right|^2 + \cdots + M \left| \frac{x}{x_0} \right|^n + \cdots$$

If $|x| < |x_0|$, then $|x/x_0| < 1$, and the progression is convergent, i.e. the series of absolute values is also convergent, which implies the (absolute) convergence of series(†). The theorem is proved.

COROLLARY. *If a power series is divergent for $x = x_0$, it is divergent for any x with an absolute value greater than that of x_0 , $|x| > |x_0|$.*

For, if it were convergent for such an x , it would be absolutely convergent by Abel's theorem for all x with absolute values less than this x , including in particular $x = x_0$, which contradicts our hypothesis.

A proposition already stated above follows from Abel's theorem and its corollary:

For every power series having both points of convergence and points of divergence, there exists a positive number R such that, for all x , $|x| < R$, the series is absolutely convergent, and for all x , $|x| > R$, the series is divergent.

As regards the values $x = R$ and $x = -R$, there are various possibilities: the series may be convergent at both points, or only at one, or at neither.

Definition. The number R such that, for all x , $|x| < R$, the power series is convergent, whilst for all x , $|x| > R$, it is divergent, is called the **radius of convergence** of the series, the interval from $x = -R$ to $x = R$ being the **interval of convergence** (it may be closed at both ends, or only at one, or may be open).

We agree to take $R = 0$ for series divergent for all x except $x = 0$, and $R = \infty$ for series convergent for all x .

II. A rule can be given for finding the radius of convergence of power series(†).

THEOREM. If $\lim \sqrt[n]{|a_n|} = \rho$ exists, the radius of convergence of the series is equal to $1/\rho$, i.e. $R = 1/\rho$, where $R = 0$ if $\rho = \infty$, and $R = \infty$ if $\rho = 0$.

Proof. We put $u_n = |a_n x^n|$, so that $u_0 + u_1 + u_2 + \dots + u_n + \dots$ is the series of absolute values of the terms of series (†):

$$|a_0| + |a_1| |x| + |a_2| |x|^2 + \dots + |a_n| |x|^n + \dots \quad (\dagger\dagger)$$

We have:

$$\sqrt[n]{u_n} = \sqrt[n]{|a_n|} |x|.$$

(1) Let ρ be a finite non-zero number. Then

$$\lim_{n \rightarrow \infty} \sqrt[n]{u_n} = \rho |x|;$$

consequently, series (††) is convergent by Cauchy's test, i.e. series (†) is absolutely convergent for $\rho |x| < 1$, i.e. for $|x| < 1/\rho$. With $\rho |x| > 1$, i.e. with $|x| > 1/\rho$, series (††) is divergent, so that series (†) cannot be absolutely convergent. But in this case it must be divergent for these values of x . For, suppose with $x = x_1$, $|x_1| > 1/\rho$, series (†) were convergent, then by Abel's theorem, it would have to be absolutely convergent for $x = x_2$, where $|x_1| > |x_2| > 1/\rho$, which is impossible, as we have seen. The series is therefore convergent for $|x| < 1/\rho$ and divergent for $|x| > 1/\rho$, i.e. $R = 1/\rho$.

(2) Let $\rho = 0$. Then $\lim_{n \rightarrow \infty} \sqrt[n]{u_n} = 0$ for any x , and series (††) is convergent for any x . Thus series (†) is absolutely convergent at every point of Ox and $R = \infty$.

(3) Let $\rho = \infty$. Then $\lim_{n \rightarrow \infty} \sqrt[n]{u_n} = \infty$ for any x , $x \neq 0$, i.e. series (†) cannot be absolutely convergent whatever the $x \neq 0$. We conclude by Abel's theorem that series (†) is divergent at every point of Ox (except zero) and $R = 0$.

The above rule for finding the radius of convergence was obtained from Cauchy's test. A similar rule can be derived from d'Alembert's test.

THEOREM. If $\lim |u_{n-1}/u_n| = \rho$, the radius of convergence of the series is equal to $1/\rho$, i.e. $R = 1/\rho$, where we take $R = 0$ with $\rho = \infty$ and $R = \infty$ with $\rho = 0$.

We notice that the rule following from the last theorem, whilst very convenient in practice, is only applicable to series with no "gaps" (as from a certain $n = N$), i.e. such that $a_k \neq 0$ ($k > N$).

The d'Alembert and Cauchy tests can always be applied directly to series(††) for finding the radius of convergence of series(†).

132. General properties of power series. We shall use the properties of functional series (art. 2) to prove three theorems regarding the continuity, infinite differentiability and analyticity of the function to which a power series is convergent.

THEOREM I. The sum of the power series

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \quad (\dagger)$$

is a continuous function in any closed interval contained in the interval of convergence.

Proof. By the theorem of Sec. 127, it is sufficient to show that the power series is uniformly convergent in any interval $[-R_1, R_1]$, where $R_1 < R$. And this simply follows from the fact that, by Abel's theorem,

$$|a_0| + |a_1| R_1 + |a_2| R_1^2 + \cdots + |a_n| R_1^n \quad (\dagger\dagger)$$

is convergent. In fact, every term of power series(†) does not exceed in absolute value the corresponding term of series(††) in the interval $[-R_1, R_1]$:

$$|a_n x^n| \leq |a_n| R_1^n$$

so that, by Weierstrass's test (Sec. 127), we can conclude that the power series is uniformly convergent in $[-R_1, R_1]$.

THEOREM II. The sum of power series(†) is a function having derivatives of any order inside the interval of convergence. These derivatives are the sums of the power series got by differentiating the given series a corresponding number of times, the radius of convergence of each "derived" series being the same as that of the initial series.

Proof. Let $f(x)$ be the sum of series (†) and let us take formally the first "derived" series

$$f'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} + \cdots$$

In order to prove that this equation holds in the interval $[-R_1, R_1]$, $R_1 < R$, we have to show by theorem II of Sec. 128 that the latter series is uniformly convergent in this interval. We show that its radius of convergence is equal to the radius of convergence R of series(†). It will then follow, as in the proof of theorem I, that the "derived" series is uniformly convergent in $[-R_1, R_1]$. Let

$|x| \leq R_1 < R_2 < R$. We have:

$$|na_n x^{n-1}| \leq n |a_n| R_1^{n-1} = \frac{n |a_n| R_2^n}{R_1} \left(\frac{R_1}{R_2} \right)^n.$$

Since the series $\sum_{n=0}^{\infty} a_n R_2^n$ is convergent (Abel's theorem), all its terms are bounded by some number M :

$$|a_n R_2^n| \leq M.$$

Then

$$|na_n x^{n-1}| \leq \frac{n |a_n| R_2^n}{R_1} \left(\frac{R_1}{R_2} \right)^n \leq n \frac{M}{R_1} q^n, \quad \text{where } q = \frac{R_1}{R_2} < 1.$$

Consequently, the terms of the series are not greater in absolute value than the corresponding terms of the series

$$\frac{M}{R_1} q + 2 \frac{M}{R_1} q^2 + \cdots + n \frac{M}{R_1} q^n + \cdots,$$

which is convergent. For, on applying d'Alembert's test, we have:

$$\frac{\frac{M}{R_1} (n+1) q^{n+1}}{\frac{M}{R_1} n q^n} = \frac{n+1}{n} q \rightarrow q < 1.$$

Therefore the "derived" series is convergent for any x , $|x| < R$.

This is what we had to prove.

The derived series cannot be convergent for any x , $|x| > R$, since otherwise the original series would be convergent for the same x (by Theorem 1 of Sec. 128), which contradicts the definition of interval of convergence.

On applying the proposition just proved to the first "derived" series, we find that

$$f''(x) = 2a_2 + 3 \cdot 2a_3 x + \cdots + n(n-1)a_n x^{n-2} + \cdots;$$

similarly,

$$f'''(x) = 3 \cdot 2a_3 + \cdots + n(n-1)(n-2)a_n x^{n-3} + \cdots,$$

and so on.

Thus a power series can be differentiated term by term any number of times in its interval of convergence.

The fact that a power series can be integrated term by term in its interval of convergence follows directly from theorem 1, Sec. 128.

For instance,

$$\int_a^x f(x) dx = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots + \frac{a_n}{n+1} x^{n+1} + \cdots,$$

where $|x| < R$.

THEOREM III. The sum of a power series is an analytic function in the interval of convergence.

Proof. We take the general form of the power series

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots.$$

We showed in theorem II that $f(x)$ is infinitely differentiable in the interval of convergence $[x_0 - R, x_0 + R]$.

Let us express the coefficients of the series in terms of the derivatives of $f(x)$. As is easily seen, we have for the n -th derivative:

$$f^{(n)}(x) = n(n-1) \cdots 2 \cdot 1 a_n + (n+1)n(n-2) \cdots 2 a_{n+1}(x - x_0) + \cdots$$

On setting $x = x_0$ here, we get:

$$f^{(n)}(x_0) = n! a_n,$$

whence

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

The coefficients of the power series are therefore the corresponding coefficients of the Taylor series for $f(x)$ at $x = x_0$, and

$$\begin{aligned} f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots \\ \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots \end{aligned}$$

A new statement of the theorem follows from this:

Any power series is the Taylor series of the function represented by it.

This means that, if a function can be represented by a power series, it can only be represented by one, viz. the Taylor series, i.e. the expansion of a function into a power series is *unique*.

4. Power Series (Continued)

133. Another method of expanding functions in Taylor's series. Up till now we have found the Taylor series of a function directly: we have evaluated the Taylor coefficients by successive differentiation

of the function itself, have built up the Taylor formula, then have proved that the remainder term tends to zero in some interval of values of the independent variable. This method of solving the problem of expanding a function involves difficulties, since investigation of the remainder term is often far from easy (for instance, this is the reason why we have not yet proved that the series for $\ln(1+x)$ and $(1+x)^m$ are convergent in the interval $(-1, 0)$).

We shall now use the general properties of power series (art. 3) to develop another method of solving the problem of expanding a function in a Taylor series; this method has a number of important advantages over the previous one, as will be clearly demonstrated by the examples below.

Let $f(x)$ be an infinitely differentiable function in the neighbourhood of the point x_0 . We write formally:

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots, \quad (\dagger)$$

where $a_0, a_1, a_2, \dots, a_n, \dots$ are as yet undetermined coefficients. Suppose that we can find these coefficients (on the assumption that the function can be expanded as a power series) on the basis of some property of $f(x)$. Further, we find the interval of convergence of the series thus formed. It is the Taylor series in this interval for the function represented by it i.e. for its sum, say $F(x)$. In order to be sure that we have now obtained the Taylor expansion of $f(x)$, it only remains to show that $F(x)$ is identical with $f(x)$.

The proof is usually as follows. We show that $F(x)$ has the property of $f(x)$, on the basis of which the coefficients of series $(\dagger)$ were evaluated. If, in addition, only one function can possess this property, we can conclude that $F(x)$ and $f(x)$ are identical, and that the series obtained is the required Taylor series.

For the sake of brevity, we shall call this second method, as just described, *the method of undetermined coefficients*. The reader will get to know the method from the examples.

We prove in the first method that the remainder term in Taylor's formula tends to zero. We prove instead, in the second method, that the Taylor series of the given function represents precisely this, and no other, function.

At first sight it may seem superfluous to show that the convergent power series whose coefficients coincide with the Taylor coefficients of the given function has this function as its sum. An example of the fact that this may not always be true is provided by the function

quoted by Cauchy:

$$f(x) = e^{-\frac{1}{x^2}} \quad \text{for } x \neq 0 \quad \text{and} \quad f(0) = 0.$$

This is infinitely differentiable throughout Ox , and all the derivatives vanish at $x = 0$. For, $f'(x) = \frac{2}{x^3} e^{-\frac{1}{x^2}}$ with $x \neq 0$, and $f'(0) = 0$, since

$$\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{e^{-\frac{1}{h^2}}}{h} = 0;$$

$f''(0) = 0$, since

$$\lim_{h \rightarrow 0} \frac{f'(h) - f'(0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{2}{h^3} e^{-\frac{1}{h^2}}}{h} = 0,$$

and so on. Thus all the Taylor coefficients of $f(x)$ vanish at $x = 0$. The corresponding Taylor series consists of terms equal to zero, i.e. is convergent, though to a function identically zero and not to $f(x)$.

The $f(x)$ defined above provides an example of an infinitely differentiable, but not analytic function, in the neighbourhood of $x = 0$.

We see that a power series may be the Taylor series of an infinite set of functions and not merely of one, though it is only convergent to one of these functions, for which it is the unique power expansion.

Examples. (1) Let us expand as a Taylor series $f(x) = e^x$. We use the property that $f'(x) = f(x)$ and $f(0) = 1$.

Let

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$$

By the second condition, $f(0) = a_0 = 1$. We have from the first condition:

$$\begin{aligned} a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots \\ = 1 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \end{aligned}$$

Since this equation must be satisfied identically, $a_1 = 1$ and

$$a_n = \frac{a_{n-1}}{n}.$$

This recurrence formula gives us consecutively

$$a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{2 \cdot 3}, \dots, a_n = \frac{1}{n!}, \dots$$

The series

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

has the whole of Ox as its interval of convergence, i.e. its radius of convergence is ∞ . For (see Sec. 131),

$$\rho = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 \quad \text{and} \quad R = \infty.$$

Thus the series obtained represents for every x the function $F(x)$, which obviously satisfies the conditions imposed: $F'(x) = F(x)$, $F(0) = 1$. But these conditions define a unique function, viz. e^x . The reader may easily verify this for himself by solving the differential equation $dF(x) = F(x) dx$ with the condition $F(0) = 1$.

Thus

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

throughout Ox .

(2) Let us deduce the expansion of $(1+x)^m$ for any m as a Taylor series in the neighbourhood of $x = 0$.

We notice that $f(x) = (1+x)^m$ satisfies the conditions: $(1+x)f'(x) = mf(x)$ and $f(0) = 1$. We shall seek the power series expressing the function subject to these conditions:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots;$$

where $a_0 = 1$, since $f(0) = 1$.

On multiplying the "derived" series by $1+x$ and equating to the original series multiplied by m , we get:

$$\begin{aligned} (1+x)(a_1 + 2a_2x + \cdots + na_nx^{n-1} + \cdots) \\ = m(1 + a_1x + a_2x^2 + a_nx^n + \cdots), \end{aligned}$$

i.e.

$$\begin{aligned} a_1 + (a_1 + 2a_2)x + \cdots + [na_n + (n+1)a_{n+1}]x^n + \cdots \\ = m + ma_1x + ma_2x^2 + \cdots + ma_nx^n + \cdots. \end{aligned}$$

On equating coefficients of like powers, we find that

$$a_1 = m, a_1 + 2a_2 = ma_1, \dots, na_n + (n+1)a_{n+1} = ma_n, \dots$$

The required coefficients are found successively from these relationships:

$$a_1 = m, \quad a_2 = \frac{a_1(m-1)}{2} = \frac{m(m-1)}{2}, \dots,$$

$$a_n = \frac{m(m-1) \dots (m-n+1)}{1.2 \dots n} \dots;$$

we have obtained the binomial coefficients.

If m is not a positive integer, the series

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \dots + \frac{m(m-1) \dots (m-n+1)}{n!}x^n - \dots$$

has $(-1, 1)$ as its interval of convergence, i.e. its radius of convergence is 1. For,

$$\rho = \lim_{n \rightarrow \infty} \frac{|m(m-1) \dots (m-n)| n!}{|m(m-1) \dots (m-n+1)| (n+1)!} = \lim_{n \rightarrow \infty} \frac{|m-n|}{n+1} = 1.$$

Our series represents in the interval $(-1, 1)$ the function $F(x)$ which satisfies $(1+x)F'(x) = mF(x)$ and $F(0) = 1$, as follows at once from the formation of the coefficients; but this is also easily proved by carrying out the necessary operations on the series. These conditions define the unique function $(1+x)^m$. For, we can re-write the first conditions as

$$\frac{du}{u} = m \frac{dx}{1+x} \quad \text{where} \quad u = f(x),$$

or

$$d(\ln u) = d[m \ln(1+x)],$$

whence

$$\ln u = \ln(1+x)^m + C,$$

where $C = 0$ since $u = 1$ with $x = 0$. Thus

$$\ln u = \ln(1+x)^m \quad \text{and} \quad u = (1+x)^m,$$

and we have finally proved the convergence of the binomial series to $(1+x)^m$ in the interval $(-1, 1)$.

(3) We can similarly find the Taylor expansion of $f(x) = \ln(1+x)$, by observing that this satisfies the conditions

$$f'(x) = \frac{1}{1+x} = 1 - x + x^2 - \dots, \quad f(0) = 0.$$

We arrive at the series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + \dots$$

In Sec. 130 we only proved the validity of this expansion in the interval $(0, 1)$. We can now easily show that it holds in $(-1, 0)$ also. We have, in fact,

$$\rho = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \quad \text{and} \quad R = 1$$

i.e. the interval of convergence is $(-1, 1)$.

(4) Let us find the Taylor expansion of $f(x) = \arctan x$ in the neighbourhood of $x = 0$. We shall start from the following properties of $f(x)$:

$$(1) f'(x) = \frac{1}{1+x^2} \quad \text{and} \quad (2) f(0) = 0.$$

We put

$$f(x) = a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

($a_0 = 0$ by condition (2)). By condition (1), we have in the interval $(-1, 1)$:

$$a_1 + 2a_2 x + \dots + na_n x^{n-1} + \dots = 1 - x^2 + x^4 - \dots$$

We find by comparing coefficients:

$$a_1 = 1, a_2 = 0, a_3 = -\frac{1}{3}, a_4 = 0, a_5 = \frac{1}{5}, \dots, a_{2n} = 0, a_{2n+1} = \frac{(-1)^n}{2n+1} \dots$$

and

$$F(x) = x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \dots + \frac{(-1)^n}{2n+1} x^{2n+1} + \dots$$

The power series on the right-hand side is convergent in the interval $(-1, 1)$, because

$$\lim_{n \rightarrow \infty} \frac{|x|^{2n+3} (2n+1)}{(2n+3) |x|^{2n+1}} = |x|^2;$$

it represents a function $F(x)$ which obviously satisfies $F'(x) = 1 - x^2 + x^4 - \dots = 1/(1+x^2)$, which in turn, with the condition $F(0) = 0$, defines a unique function, viz. $\arctan x$.

The series is also convergent at $x = 1$, attaining the value $\arctan 1 = \pi/4$, in view of which:

$$\pi = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \dots + \frac{(-1)^n}{2n+1} + \dots \right).$$

This remarkable representation of the number π as an infinite series was known to Leibniz.

Thus

$$\arctan x = x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \dots + \frac{(-1)^n}{2n+1} x^{2n+1} + \dots$$

in the closed interval $[-1, 1]$.

(5) We can similarly obtain a series for $f(x) = \arcsin x$. Bearing in mind that

$$f'(x) = \frac{1}{\sqrt{1-x^2}}, \quad f(0) = 0,$$

and putting

$$f(x) = a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots,$$

we write (see Sec. 130):

$$\begin{aligned} & a_1 + 2a_2 x + \cdots + na_n x^{n-1} + \cdots \\ &= 1 + \frac{1}{2} x^2 + \frac{1 \cdot 3}{2 \cdot 4} x^4 + \cdots + \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} x^{2n} + \cdots, \end{aligned}$$

whence

$$a_{2n} = 0, \quad a_{2n+1} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot \frac{1}{2n+1}.$$

The series thus obtained has a radius of convergence 1; it is also convergent at the ends $x = -1$ and $x = 1$ of the interval. Its sum satisfies the initial conditions, which define a unique function, viz. $\arcsin x$.

Thus,

$$\begin{aligned} \arcsin x &= x + \frac{1}{2} \cdot \frac{1}{3} x^3 + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5} x^5 + \cdots + \\ &+ \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot \frac{1}{2n+1} x^{2n+1} + \cdots \end{aligned}$$

in the closed interval $[-1, 1]$.

134. Some applications of Taylor's series.

I. APPROXIMATE EVALUATION OF FUNCTIONS. Suppose that $f(x)$ is an analytic function in an interval $[a, b]$ and that we know the value of $f(x)$ and its successive derivatives at some point $x = x_0$ of the interval. The exact value of $f(x)$ at some other point of the interval can now be evaluated from the Taylor series, and its approximate value from a partial sum of this series or from the corresponding Taylor formula.

We have already spoken in Sec. 79 of the application of Taylor's formula for evaluating a function. The use of partial sums of the Taylor series is usually more convenient for this purpose, for two reasons.

(1) When a function is replaced by its Taylor polynomial it is sometimes possible to give a more accurate estimate of the error if we start from the entire series instead of from the Taylor formula; in other words, the maximum error due to taking the first terms of the Taylor series is smaller than the estimate of the remainder term of the corresponding Taylor formula.

(2) The expression for the given function as a series can sometimes be transformed very simply into another series, which is "more rapidly convergent".

Definition. One series is said to converge "more rapidly" or "better" than another, if we can achieve the same accuracy by confining ourselves to fewer initial terms in the first series than in the second. The transformation of a series into one which converges better is called improving the convergence of the series.

(1) We illustrate the first of the above points by taking the Taylor formula and Taylor series for the number e . We found in Sec. 79 that

$$e \approx 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!},$$

and that the maximum error δ_n is equal to $3/(n+1)!$

If we bear in mind the whole of the series for e , the error of the approximate value deduced above will be equal to the remainder R_n of the series, where

$$\begin{aligned} R_n &= \frac{1}{(n+1)!} + \frac{2}{(n+2)!} + \cdots = \frac{1}{(n+1)!} \left[1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots \right] \\ &< \frac{1}{(n+1)!} \left[1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots \right] = \frac{1}{(n+1)!} \frac{1}{1 - \frac{1}{n+1}} = \frac{1}{n!n}. \end{aligned}$$

We can thus take $\delta_n = 1/n!n$, which gives a better estimate of the error than $\delta_n = 3/(n+1)!$

Suppose, for instance, we are asked to calculate e to an accuracy of 0.01. We find from

$$\delta_n = \frac{1}{n!n} \leq \frac{1}{100}, \quad \text{i.e.} \quad (n!n) \geq 100,$$

that n can be taken equal to 4 (we need not be disturbed by the fact that the left-hand side is now just slightly below 100, since our estimate of R_n is clearly exaggerated; if we estimate R_n more accurately:

$$R_n < \frac{1}{(n+1)!} \left[1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \cdots \right] = \frac{n+2}{(n+1)!(n+1)},$$

the right-hand side of this expression is precisely 0.01 with $n = 4$). We are thus justified in asserting that $1 + 1 + 1/2! + 1/3! + 1/4! \approx 2.71$ differs by not more than 0.01 from e .

We now take δ_n from Taylor's formula. It follows from this that n must be taken so that

$$\frac{3}{(n+1)!} \leq \frac{1}{100}, \quad \text{i.e.} \quad (n+1)! \geq 300,$$

for which n must be not less than 5. Hence, when Taylor's formula is used, we can only be sure of achieving an accuracy of 0.01 when we take $n = 5$. Whereas, as we have seen, this accuracy is obtained with $n = 4$ by using the Taylor series.

(2) We illustrate the second of the above points by taking the familiar series

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots, \quad -1 < x \leq 1.$$

This series is very slowly convergent. It follows from Leibniz's theorem on alternating series that we have to take no less than the first 100,000 terms in order to calculate $\ln(1+x)$ to an accuracy of 0.00001 (say with $x=1$). A similar result is found from an estimate of the remainder term in Taylor's formula for $\ln(1+x)$. This summation is not practicable.

The speed of convergence of the series can be improved. On replacing x by $-x$ and subtracting from the expression for $\ln(1+x)$, we get

$$\ln \frac{1+x}{1-x} = 2 \left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots \right), \quad -1 < x < 1.$$

The series on the right-hand side is in itself more rapidly convergent than the series for $\ln(1+x)$. Moreover, the logarithms of any (positive) numbers can be calculated from this formula. For, when x varies in the interval of convergence of the series $(-1, 1)$, the continuous function $\frac{1+x}{1-x}$ runs through the entire interval $(0, \infty)$. Let us work out $\ln 2$ from the formula. If $\frac{1+x}{1-x} = 2$, then $x = 1/3$. We take the n -th partial sum:

$$\ln 2 \approx 2 \left(\frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3^3} + \dots + \frac{1}{2n+1} \cdot \frac{1}{3^{n+1}} \right).$$

We estimate the error thus:

$$\begin{aligned} & 2 \left(\frac{1}{2n+3} \cdot \frac{1}{3^{2n+3}} + \frac{1}{2n+5} \cdot \frac{1}{3^{2n+5}} + \dots \right) \\ & < \frac{2}{(2n+3)3^{2n+3}} \left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \dots \right) = \frac{2 \cdot 9}{(2n+3)3^{2n+3} \cdot 8}. \end{aligned}$$

We find the n for which the error does not exceed 0.00001. This must be with

$$4(2n+3)3^{2n+1} \geq 10^5,$$

which certainly holds for $n > 4$. Thus

$$\ln 2 \approx 2 \left(\frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3^3} + \frac{1}{5} \cdot \frac{1}{3^5} + \frac{1}{7} \cdot \frac{1}{3^7} + \frac{1}{9} \cdot \frac{1}{3^9} \right) = 0.693144.$$

It is thus sufficient to take the first five terms in the new series in order to find the result to the same accuracy as when the first 100,000 terms are taken in the old series. (It can easily be shown, by the way, that we need actually only take four terms.)

On setting $x = 1/(2N+1)$ in the series obtained, where N is a positive integer, we arrive at the formula

$$\ln \frac{N}{N+1} = \ln(N+1) - \ln N = 2 \left(\frac{1}{2N+1} + \frac{1}{3} \cdot \frac{1}{(2N+1)^3} + \dots \right),$$

which in fact serves for calculating the logarithms of the integers one after another, to any accuracy that may be required in practice.

II. SERIES EXPANSION OF IMPLICIT FUNCTIONS. The method of "undetermined coefficients", which we have used for finding the Taylor series of a function from its characteristic property (Sec. 133), can also be used when this property is given in the form of finite (and not differential, as in Sec. 133) relationships.

In other words, the method can be applied for finding an explicit expression (in the form of a series) for a function given implicitly by an equation between the independent variable and the function.

We take as an example the equation (cf. Sec. 13):

$$xy - e^x + e^y = 0,$$

giving y implicitly as a function of x . A general theorem exists, on the basis of which we can conclude that y is an analytic function of x in a neighbourhood of $x = 0$.

Let us find the first terms of its expansion as a Taylor series at $x_0 = 0$. We write:

$$y = a_1 x + a_2 x^2 + a_3 x^3 + \dots;$$

$a_0 = 0$, since it follows from the equation that $y = 0$ for $x = 0$. We find the undetermined coefficients $a_1, a_2, a_3, \dots$ from the condition that the series satisfies the given equation. We have:

$$\begin{aligned} x(a_1 x + a_2 x^2 + a_3 x^3 + \dots) - \left(1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \dots\right) + \\ + \left[1 + (a_1 x + a_2 x^2 + a_3 x^3 + \dots) + \frac{1}{2!} (a_1 x + a_2 x^2 + a_3 x^3 + \dots)^2 + \right. \\ \left. + \frac{1}{3!} (a_1 x + a_2 x^2 + a_3 x^3 + \dots)^3 + \dots\right] = 0 \end{aligned}$$

or

$$\begin{aligned} (a_1 x^2 + a_2 x^3 + \dots) - \left(1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \dots\right) + \left[1 + (a_1 x + a_2 x^2 + \right. \\ \left. + a_3 x^3 + \dots) + \frac{1}{2!} (a_1^2 x^2 + 2a_1 a_2 x^3 + \dots) + \frac{1}{3!} (a_1^3 x^3 + \dots) + \dots\right] = 0, \end{aligned}$$

whence

$$\begin{aligned} (-1 + a_1)x + \left(a_1 - \frac{1}{2!} + a_2 + \frac{1}{2!} a_1^2\right)x^2 + \\ + \left(a_2 - \frac{1}{3!} + a_3 + a_1 a_2 + \frac{1}{3!} a_1^3\right)x^3 + \dots = 0. \end{aligned}$$

This power series is (uniformly and absolutely) convergent in a neighbourhood of $x = 0$, at which point it becomes a power expansion of zero. The coefficients of all powers of x must therefore vanish:

$$-1 + a_1 = 0,$$

$$a_1 - \frac{1}{2} + a_2 + \frac{1}{2} a_1^2 = 0,$$

$$a_2 - \frac{1}{6} + a_3 + a_1 a_2 + \frac{1}{6} a_1^3 = 0.$$

We find from this system:

$$\begin{aligned} \text{i.e.} \quad & a_1 = 1, \quad a_2 = -1, \quad a_3 = 2; \\ & y = x - x^2 + 2x^3 + \dots \end{aligned}$$

We have obtained the first piece of the Taylor series for our function; this is an entirely satisfactory representation of the function from the point of view of accuracy in a small enough neighbourhood of $x = 0$.

We remark finally that the coefficients of the Taylor series in the present problem (and in other similar problems) can be found by using the expressions for the coefficients in terms of derivatives of the function. The values of the latter at the point in question (here $x = 0$) are calculated by successive differentiation of the equation for the function. The reader may easily verify from the present example that the same result is obtained.

III. INTEGRATION OF FUNCTIONS. Suppose that we want to find

$$F(x) = \int_a^x f(x) dx,$$

the expansion of the integrand $f(x)$ as a Taylor series being known; the limits of the integral lie inside the interval of convergence of the series. We are now justified in integrating the series term by term. As a result a Taylor series is obtained for the function $F(x)$ with the same interval of convergence. If $F(x)$ can be expressed by an elementary function (in a finite form), we arrive by this means at the Taylor expansion of the elementary function. Whereas if the finite expression for $F(x)$ is unknown or does not exist (see Sec. 105, II), the series obtained can serve as an expression for $F(x)$ in terms of the simplest of all the basic elementary functions—the power functions—though admittedly in an infinite, not in a finite, form. However, since the properties established for a power series in its interval of convergence are completely analogous to the properties of a finite expression, such a representation of a function with the aid of an infinite series can be regarded as no “worse” than a representation with the aid of a finite number of basic elementary functions. On the contrary, by virtue of the simplicity of the terms, the series is much more convenient in many respects than a function which is not a power function, and attempts are often made to write a familiar elementary or even basic elementary function as an infinite power series.

Examples. (1) We take the integral

$$F(x) = \int_0^x \frac{dx}{\sqrt{1-x^2}}.$$

We have in accordance with Sec. 130:

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}x^8 + \dots;$$

bearing in mind that $F(x) = \arcsin x$, we find on integrating the series term by term:

$$\arcsin x = x + \frac{1}{2} \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{x^7}{7} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \frac{x^9}{9} + \dots$$

The expansions of $\ln(1+x)$ and $\arctan x$ are easily obtained in a similar manner:

$$\ln(1+x) = \int_0^x \frac{dx}{1+x} = \int_0^x (1-x+x^2-\dots) dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots,$$

$$\arctan x = \int_0^x \frac{dx}{1+x^2} = \int_0^x (1-x^2+x^4-\dots) dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

The present method of expanding $\arcsin x$, $\ln(1+x)$, $\arctan x$, at bottom differs only in form from the "method of undetermined coefficients", with the aid of which these functions were expanded in Sec. 133.

It may be mentioned that, if an estimate is known for the remainder term of the series for the integrand $f(x)$, we can also estimate the remainder term of the series for the integral $F(x)$, on the basis of the theorem on estimating an integral.

(2) Suppose we are given

$$\int \frac{\sin x}{x} dx$$

(which is not expressible in the finite form, see Sec. 105). On dividing the series for $\sin x$ by x , we get:

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \dots$$

This series, like the series for $\sin x$, has the whole of Ox as its interval of convergence. Integration gives:

$$\int \frac{\sin x}{x} dx = C + x - \frac{x^3}{3!3} + \frac{x^5}{5!5} - \dots$$

This series is not convergent to any elementary function; it is the analytic expression of a new function, though by means of an infinite, and not a finite, number of operations.

(3) Similarly, the elliptic integral (not expressible in a finite form) can be represented in terms of elementary functions. Let us take, say, the problem of finding the length L of an ellipse with semi-axes a and b . We have (Sec. 118):

$$L = 4a \int_0^{\pi/2} \sqrt{1 - \varepsilon^2 \cos^2 t} dt,$$

where ε is the eccentricity of the ellipse.

Since $\varepsilon < 1$, $\varepsilon^2 \cos^2 t < 1$, and the integrand can be expanded in a binomial series (see Sec. 130):

$$\sqrt{1 - \varepsilon^2 \cos^2 t} = 1 - \frac{1}{2} \varepsilon^2 \cos^2 t - \frac{1}{2 \cdot 4} \varepsilon^4 \cos^4 t - \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} \varepsilon^6 \cos^6 t - \dots$$

Integration gives:

$$\begin{aligned} L &= 2\pi a - 4a \left(\frac{1}{2} \varepsilon^2 \int_0^{\pi/2} \cos^2 t \, dt + \frac{1}{2 \cdot 4} \varepsilon^4 \int_0^{\pi/2} \cos^4 t \, dt + \right. \\ &\quad \left. + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} \varepsilon^6 \int_0^{\pi/2} \cos^6 t \, dt + \dots \right) \\ &= 2\pi a \left(1 - \frac{1}{2 \cdot 2} \varepsilon^2 - \frac{1 \cdot 3}{2 \cdot 4 \cdot 8} \varepsilon^4 - \frac{1 \cdot 3 \cdot 15}{2 \cdot 4 \cdot 6 \cdot 48} \varepsilon^6 - \dots \right). \end{aligned}$$

This series provides an expansion of the length of the ellipse in powers of the eccentricity.

Confining ourselves to the terms written, the remainder of the series for the integrand:

$$-\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \varepsilon^8 \cos^8 t + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \varepsilon^{10} \cos^{10} t + \dots \right)$$

will not exceed in absolute value the expression

$$\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \varepsilon^8 \cos^8 t (1 + \varepsilon^2 + \varepsilon^4 + \dots) = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \varepsilon^8 \cos^8 t \cdot \frac{1}{1 - \varepsilon^2}.$$

The remainder of the series for L will therefore not exceed

$$\begin{aligned} 4a \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \frac{\varepsilon^8}{1 - \varepsilon^2} \int_0^{\pi/2} \cos^8 t \, dt &= 4a \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \frac{\varepsilon^8}{1 - \varepsilon^2} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \frac{\pi}{2} \\ &= \frac{175}{64 \cdot 128} \frac{\varepsilon^8}{1 - \varepsilon^2} \pi a. \end{aligned}$$

Consequently,

$$L = 2\pi a \left(1 - \frac{1}{4} \varepsilon^2 - \frac{3}{64} \varepsilon^4 - \frac{5}{256} \varepsilon^6 \right) + R,$$

where $|R| < 0.022 \frac{\varepsilon^8}{1 - \varepsilon^2} \pi a$. This formula can be used to find the length with sufficient accuracy when ε is small.

135. Functions of a complex variable. Euler's formulae. Use is generally made only of real functions of real variables in an ordinary course of mathematical analysis. Certain operations, however (e.g. the solution of equations), when carried out in the domain of real numbers, lead to complex numbers. An attempt is made in such cases to free the final results of complex numbers, in order not to depart from the domain of real numbers.

We assume that the reader is acquainted with the algebra of complex numbers from a course of elementary mathematics, and merely recall briefly the main propositions.

Let $z = x + iy$ be a given complex number, where x and y are real numbers and $i^2 = -1$; x is called the real part, and iy the imaginary part of z . The number $x - iy$ is called the conjugate of z ; it is written as $\bar{z}$. The modulus of $z = x + iy$ is defined as $|z| = \sqrt{x^2 + y^2}$. The equation $z = x + yi = 0$ can only hold if $|z| = 0$, i.e. if both $x = 0$ and $y = 0$. A complex number is represented geometrically by a point on a plane provided with a system of Cartesian co-ordinates, the abscissa of the point being the real, and the ordinate the imaginary part of the number. The axis of abscissae is therefore called the real axis, and the axis of ordinates the imaginary axis; the plane is called the complex plane. If a point represents a number z , it is called "the point z ". The radius vector of the point can also be regarded as representing the complex number (its projections on the axes are the real and imaginary parts of the number). The length of the radius vector is clearly equal to the modulus of the number. The point $\bar{z}$ is symmetrical to the point z about the real axis (i.e. axis of abscissae).

If we pass to polar co-ordinates ρ, φ by the usual formulae, we obviously have

$$z = x + iy = \rho(\cos \varphi + i \sin \varphi), \quad (\dagger)$$

where ρ is the modulus of z , i.e.

$$|z| = \rho = \sqrt{x^2 + y^2},$$

and φ is called the argument of z , i.e.

$$\arg z = \varphi = \arctan \frac{y}{x}.$$

When a complex number is given, its argument is not specified uniquely, but only to an integral number of complete revolutions. The right-hand side of $(\dagger)$ is called the trigonometric form of the complex number.

If $y = 0$, the complex number $z = x$ is a real number, represented by a point of the real axis, whilst if $x = 0$, $z = iy$ is called an *imaginary number*, represented by a point of the imaginary axis.

Since real numbers form part of the set of all complex numbers, the rules for the arithmetical operations on complex numbers are established so as to preserve all the properties of operations on real numbers (except for properties following from the concepts of "greater" or "less" than, which have no meaning for non-real complex numbers).

The operations are carried out in accordance with the formulae:

$$(1) \quad (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$

i.e. the sum of two complex numbers is a complex number, the real and imaginary parts of which are the sums of the corresponding parts of the terms. This rule corresponds geometrically to the familiar rule of vector algebra for geometrical addition of vectors.

$$(2) \quad (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2);$$

subtraction is the inverse operation to addition, and expresses analytically the usual geometrical subtraction of vectors. The difference between two complex numbers $z_1 - z_2$ is represented by a vector with initial point at z_2 and end at z_1 . Thus $|z_1 - z_2|$ is the length of the straight segment joining z_1 and z_2 .

In accordance with the subtraction rule, we conclude that two complex numbers can be equal when and only when their real and imaginary parts are equal.

$$(3) \quad (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1);$$

this rule does not correspond to either of those for multiplication of vectors. If the factors are taken in the trigonometric form, it is readily deduced that

$$\rho_1(\cos\varphi_1 + i\sin\varphi_1) \cdot \rho_2(\cos\varphi_2 + i\sin\varphi_2) = \rho_1\rho_2[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)],$$

i.e. *multiplication of two numbers implies multiplying their moduli and adding their arguments.*

On successively extending to any number n of factors and taking them equal, we arrive at the rule for raising a complex number to an integral power:

$$[\rho(\cos\varphi + i\sin\varphi)]^n = \rho^n(\cos n\varphi + i\sin n\varphi),$$

n is a positive integer.

This is known as de Moivre's formulae.

Let us multiply a complex number by its conjugate: the product is a positive real number equal to the square of the modulus of each factor:

$$(4) \quad \begin{aligned} z\bar{z} &= (x + iy)(x - iy) = x^2 + y^2 = |z|^2 = |\bar{z}|^2. \\ \frac{x_1 + iy_1}{x_2 + iy_2} &= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i \frac{-x_1y_2 + x_2y_1}{x_2^2 + y_2^2} \end{aligned}$$

on condition that $x_2 + iy_2 \neq 0$. Division is the inverse operation to multiplication. If the divisor and dividend are taken in the trigonometric form, we have

$$\frac{\rho_1(\cos\varphi_1 + i\sin\varphi_1)}{\rho_2(\cos\varphi_2 + i\sin\varphi_2)} = \frac{\rho_1}{\rho_2}[\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)],$$

i.e. *on division, the modulus of the dividend is divided by the modulus of the divisor whilst the argument of the divisor is subtracted from the argument of the dividend.*

Careful consideration of the above formulae shows that the rules expressed by them can be combined into a single, very simple rule:

The arithmetical operations are carried out on complex numbers in accordance with the usual rules, applied to the numbers as though they were ordinary binomials $(a + ib)$, but with i^2 always replaced in the result by -1 .

The laws and properties of the arithmetical operations, such as are familiar in the domain of real numbers, are carried over without change to the domain of complex numbers.

Notice that, if all the numbers in an arithmetical operation are replaced by their conjugates, the result is the conjugate of the original result. For, on changing the signs of y_1 and y_2 , the signs of the imaginary parts of the right-hand sides of all the formulae (1), (2), (3), (4) are changed. But if this is true in each of the four arithmetical operations, it must be true for any combination of them.

The complex quantity $z = x + iy$ is called a variable if it takes different numerical (complex) values. We say that *the complex quantity $w = u + iv$ is a function of the complex independent variable $z = x + iy$, if for each value of this variable belonging to some set of complex numbers, called the domain of variation of the function, there are one or more corresponding definite values of the quantity w* . We usually write $w = f(z)$.

The fundamental concepts of mathematical analysis can be introduced for functions of a complex variable (limit, continuity, derivative, integral, series etc.), in complete analogy with the corresponding concepts for functions of a real variable. We must leave this on one side, since it would require a special exposition going beyond the limits of this book. We shall make use of any necessary simple considerations without providing a strict basis.

Various methods can be used for defining the basic elementary functions of a complex variable; but since we already know the power function $w = z^n$ (with n a positive integer), the simplest of all is by means of a power series.

It can be shown that *the series with complex terms $w_1 + w_2 + \dots + w_n + \dots$, where $w_n = u_n + iv_n$, u_n and v_n are real numbers or functions, is convergent if the series (with real terms), consisting of the moduli of the original terms, is convergent*. The series is then said to be absolutely convergent, its sum s being equal to $u + iv$, where

$$u = u_1 + u_2 + \dots + u_n + \dots, \quad v = v_1 + v_2 + \dots + v_n + \dots$$

The statement of Abel's theorem of Sec. 131 holds for the power series of a complex variable z :

$$a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots,$$

where $a_0, a_1, a_2, \dots, a_n, \dots$ are complex constants; the existence follows from this of a number R such that, for all z satisfying $|z| < R$, the series is absolutely convergent, and is divergent for all $|z| > R$. If $|z| < R$, the point z lies in the circle with centre at the origin and radius R ; R is in fact called the radius of convergence of the series, whilst $|z| < R$ is the circle of convergence. The radius of convergence may be zero or infinity, as well as finite; in the first case the series is convergent at a single point, and in the second at any point of the plane.

The theory of power series discussed in the previous article holds without any essential modifications for power series in the complex domain.

By virtue of this theory, a power series is absolutely and uniformly convergent in its circle of convergence to a continuous and infinitely differentiable function. The function is said to be analytic in this circle.

The series

$$1 + \frac{1}{1!} z + \frac{1}{2!} z^2 + \cdots + \frac{1}{n!} z^n + \cdots$$

is convergent for any z , i.e. throughout the complex plane. At points of the real axis, when $z = x$, this series represents the function e^x ; it is natural to assume that the series represents a function which we shall describe as exponential (to base e) and write as e^z for any complex z . Thus we have, by definition,

$$e^z = 1 + \frac{1}{1!} z + \frac{1}{2!} z^2 + \cdots + \frac{1}{n!} z^n + \cdots$$

In view of this definition and the invariance of the rules for operations on numbers and series when passing to the complex plane, the properties of the function e^x which follow from these operations remain valid for e^z also. For instance, $e^{z_1 z_2} = e^{z_1} \cdot e^{z_2}$.

If $y = 0$, i.e. $z = x$, e^z is the familiar exponential function e^x . We now consider how the function e^z can be expressed if $x = 0$, i.e. $z = iy$, where y is a real number. We have

$$e^{iy} = 1 + \frac{1}{1!} iy + \frac{1}{2!} (iy)^2 + \frac{1}{3!} (iy)^3 + \cdots + \frac{1}{n!} (iy)^n + \cdots,$$

i.e.

$$e^{iy} = 1 + i \frac{1}{1!} y - \frac{1}{2!} y^2 - i \frac{1}{3!} y^3 + \frac{1}{4!} y^4 + i \frac{1}{5!} y^5 - \cdots$$

or

$$e^{iy} = \left(1 - \frac{1}{2!} y^2 + \frac{1}{4!} y^4 - \frac{1}{6!} y^6 + \cdots \right) + \\ + i \left(y - \frac{1}{3!} y^3 + \frac{1}{5!} y^5 - \frac{1}{7!} y^7 + \cdots \right).$$

But the series in brackets represent respectively $\cos y$ and $\sin y$. Therefore $e^{iy} = \cos y + i \sin y$. In order to underline the generality of the result, we replace the notation for the variable y by t :

$$e^{it} = \cos t + i \sin t. \quad (*)$$

This is Euler's formula, expressing the exponential in terms of the trigonometric functions. By virtue of this formula, every complex

number can be written in the so-called exponential form:

$$z = \rho(\cos \varphi + i \sin \varphi) = \rho e^{i\varphi},$$

where $\rho = |z|$, $\varphi = \arg z$. We notice that, when φ varies from 0 to 2π , the point $e^{i\varphi} = \cos \varphi + i \sin \varphi$ runs once round the circle with centre at the origin and radius 1.

On replacing t by $-t$, we get:

$$e^{-it} = \cos t - i \sin t. \quad (**)$$

We find from the two equations(*) and(**):

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \quad \sin t = \frac{e^{it} - e^{-it}}{2i}.$$

These are also Euler's formulae; they express the trigonometric functions in terms of the exponential.

We shall later need to differentiate the function $y = e^{rx}$ with respect to x , where x is a real variable and $r = \alpha + i\beta$ is a complex number. It is easily shown that the differentiation is accomplished as though r were real; in fact,

$$\frac{d}{dx} e^{rx} = r e^{rx}.$$

We have $y = e^{rx} = e^{\alpha x} \cdot e^{i\beta x}$, i.e. by Euler's formula,

$$y = e^{\alpha x}(\cos \beta x + i \sin \beta x).$$

On differentiating the sum by the usual rule, we get

$$\begin{aligned} y' &= \alpha e^{\alpha x}(\cos \beta x + i \sin \beta x) + \beta e^{\alpha x}(-\sin \beta x + i \cos \beta x) \\ &= \alpha e^{\alpha x}(\cos \beta x + i \sin \beta x) + i\beta e^{\alpha x}(i \sin \beta x + \cos \beta x) \\ &= (\alpha + i\beta) e^{\alpha x}(\cos \beta x + i \sin \beta x) = r e^{rx}. \end{aligned}$$

This is what we wished to prove.

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